# Sum rules in the heavy quark limit of QCD

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In the leading order of the heavy quark expansion, we propose a method within the operator product expansion and the trace formalism that allows us to obtain, in a systematic way, Bjorken-like sum rules for the derivatives of the elastic Isgur-Wise (IW) function $\xi(w)$ in terms of corresponding Isgur-Wise functions of transitions to excited states. A key element is the consideration of the nonforward amplitude, as introduced by Uraltsev. A simplifying feature of our method is to consider currents aligned along the initial and final four-velocities. As an illustration, we give a very simple derivation of Bjorken and Uraltsev sum rules. On the other hand, we obtain a new class of sum rules that involve the products of IW functions at zero recoil and IW functions at any *w*. Special care is given to the needed derivation of the projector on the polarization tensors of particles of arbitrary integer spin. The new sum rules give further information on the slope $\rho^2=-\xi'(1)$ and also on the curvature $\sigma^2=\xi''(1)$ and imply, modulo a very natural assumption, the inequality $\sigma^2 \geqslant (5/4)\rho^2$, and therefore the absolute bound $\sigma^2 \geqslant 15/16$.

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## I. INTRODUCTION

Since the formulation of the Bjorken sum rule [1], other sum rules (SRs) have been derived involving leading and subleading quantities in heavy quark expansion [2–6]. The recent Uraltsev SR [6,7] at leading order came as a big surprise, leading to a rigorous lower bound for the elastic Isgur-Wise (IW) function[1] $\rho^2 \geqslant 3/4$. As with earlier SRs, one gets the impression that these results come out like fishing in a lake, swarming with sum rules, the success of the catch depending on the genius or skill of the particular authors. Hence we need to have a systematic way of formulating these SRs. This is the subject of the present paper, although only in the particular case of IW functions in the heavy quark limit of QCD. The method can be easily applied to subleading form factors [8].

In the derivation of the sum rules we will make use of the operator product expansion (OPE) [9] in heavy quark transitions [2,5,6,10], in a manifestly covariant approach.

To be completely general, let us consider the direct graphs

$$B_i(v_i) \stackrel{\Gamma_1}{\longrightarrow} D^{(n)}(v') \stackrel{\Gamma_2}{\longrightarrow} B_f(v_f),$$

where $B_i$ and $B_f$ are ground state $B$ or $B^*$ mesons and $D^{(n)}$ are all possible ground state or excited $D$ mesons coupled to $B_i$ and $B_f$ through the currents $\overline{h}_c(v')\Gamma_1 h_b(v_i)$ and $\overline{h}_b(v_f)\Gamma_2 h_c(v')$. The Dirac matrices $\Gamma_i$ $(i=1,2)$ are arbitrary and can be chosen to derive relations involving definite current matrix elements.

Let us summarize the general argument. We consider two arbitrary currents:

$$J_1(x)=\overline{c}(x)\Gamma_1 b(x), \quad J_2(y)=\overline{b}(y)\Gamma_2 c(y) \tag{1}$$

and their $T$ product:

$$T_{fi}(q)=i\int d^4x\, e^{-iq\cdot x}\langle B_f|T[J_2(x)J_1(0)]|B_i\rangle. \tag{2}$$

As explained in detail, for example, in Ref. [5], on inserting in this expression intermediate states, $x<0$ receives contributions from the direct channel with a single heavy quark $c$, while $x>0$ receives contributions from intermediate states with $b\overline{c}b$ quarks, the $Z$ diagrams. The energy denominators are $M_B-q^0-E_{X_c}$ for the direct graphs and $M_B+q^0-(E_{X'_c}+2M_B)$ for the $Z$ diagrams. Taking the typical virtuality of the direct channels $V=M_B-q^0-E_{X_c}$ such that $\Lambda_{\mathrm{QCD}} \ll V \ll M_B$, one sees that the direct channels contribute at the order $1/V$ and the $Z$ diagrams at the order $1/(-V-2M_D)$. In both cases the absolute value of the denominator is $\gg \Lambda_{\mathrm{QCD}}$. This allows one to approximate Eq. (2) with the leading contribution to the OPE [10]:

$$T_{fi}(q)=i\int d^4x\, e^{-iq\cdot x}\langle B_f|\overline{b}(x)\Gamma_2 S_c(x,0)\Gamma_1 b(0)|B_1\rangle + O(1/m_c^2), \tag{3}$$

where $S_c(x,0)$ is the free charm quark propagator if $O(\alpha_s)$ corrections are neglected. The $c$ quark propagator has two terms, a positive energy denominator $\sim V$ and a negative energy denominator $\sim(-V-2m_c)$. Varying $V$ independently of $m_c$ one can equate the direct channel contribution to Eq. (2) to that of the positive energy pole of the $c$ quark propagator in Eq. (3), the so-called OPE side, giving the following result that involves only the direct channel:

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[1] This bound was obtained in a class of relativistic quark models [11,12] that were afterwards shown to satisfy the Uraltsev sum rule [13].

 

$$\frac{1}{2v'^0\sqrt{4v_i^0 v_f^0}}\Bigg\{\sum_{D=P,V}\sum_n \mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\overline{\Gamma}_2\mathcal{D}^{(n)}(v')]\mathrm{Tr}[\overline{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_1(v_i)]\xi^{(n)}(w_i)\xi^{(n)*}(w_f) + \text{contribution from other excited states} + O(1/m_Q)\Bigg\}$$

$$= -\frac{1}{\sqrt{4v_i^0 v_f^0}}\xi(w_{if})\mathrm{Tr}\left[\overline{\mathcal{B}_f}(v_f)\Gamma_2\frac{\not{v}'_c+1}{2v_c'^0}\Gamma_1\mathcal{B}_1(v_i)\right]+O(1/m_Q). \tag{4}$$

In this equation, $(\not{v}'_c+1)/2v_c'^0$ is the positive energy residue of the $c$ quark propagator and the left-hand side (LHS) is the sum over all possible ground state or excited $D$ mesons. We have adopted the trace formalism for the current matrix elements [4,14] and made explicit in Eq. (4) the sum over pseudoscalar and vector $D$ ($D^*$) mesons and their radial quantum numbers. In relation (4),

$$w_i = v_i\cdot v', \quad w_f = v_f\cdot v', \quad w_{if} = v_i\cdot v_f. \tag{5}$$

In the LHS there are also leading order contributions of excited states and subleading terms coming from the ground state or from transitions between the ground state and excited states, denoted by $O(1/m_Q)$, where $m_Q$ can be $m_c$ or $m_b$.

One main point we want to emphasize is that in the OPE side the ground state IW function $\xi(w_{if})$ appears since, following Uraltsev [6], we assume in general $v_i\neq v_f$ and take $B_i$ and $B_f$ to be ground state $B$ mesons. Of course, for $w_{if}=1$ one gets $\xi(1)=1$, $w_i=w_f=w$, and the general formula (4) takes the more familiar form [5]

$$\frac{1}{4v^0v'^0}\Bigg\{\sum_{D=P,V}\sum_n \mathrm{Tr}[\overline{\mathcal{B}_f}(v)\overline{\Gamma}_2\mathcal{D}^{(n)}(v')] \times\mathrm{Tr}[\overline{\mathcal{D}}^{(n)}(v')\Gamma_1\mathcal{B}_i(v)]|\xi^{(n)}(w)|^2 + \text{contribution from other excited states} + O(1/m_Q)\Bigg\}$$

$$= -\frac{1}{2v^0}\mathrm{Tr}\left[\overline{\mathcal{B}_f}(v)\Gamma_2\frac{\not{v}'_c+1}{2v_c'^0}\Gamma_1\mathcal{B}_1(v)\right]+O(1/m_Q^2). \tag{6}$$

But let us keep to the general case $v_i\neq v_f$. By choosing in a convenient way the initial and final mesons $B_i$ and $B_f$ and the Dirac matrices $\Gamma_1$ and $\Gamma_2$, one can derive sum rules at the leading order (the Bjorken SR [1] and the Uraltsev SR [6]) and also SRs involving subleading Isgur-Wise functions, as we obtained in Ref. [5]. To illustrate the method, we will limit ourselves in this paper to the heavy quark limit.

In the heavy quark limit, since we can make the four-velocity of the intermediate quark equal to the intermediate hadron velocity, $v'_c=v'$, relation (4) is written, multiplying by $2v'_0\sqrt{4v_i^0v_f^0}$,

$$L(w_{if},w_i,w_f)=R(w_{if},w_i,w_f), \tag{7}$$

where $L(w_{if},w_i,w_f)$ stands for the LHS [the sum over intermediate states $\mathcal{D}^{(n)}(v')$] and $R(w_{if},w_i,w_f)$ stands for the RHS [the OPE side, proportional to $\xi(w_{if})$].

The variables $w_{if}$, $w_i$, and $w_f$ are independent within a certain domain. Indeed, without loss of generality one can take

$$v_i=(1,0,0,0), \quad v_f=(\sqrt{1+a^2},0,0,a),$$
$$v'=(\sqrt{1+b^2+c^2},0,b,c), \tag{8}$$

giving

$$w_{if}=\sqrt{1+a^2}, \quad w_i=\sqrt{1+b^2+c^2},$$
$$w_f=\sqrt{1+b^2+c^2}\sqrt{1+a^2}-ac. \tag{9}$$

One has three independent parameters $a$, $b$, and $c$ or equivalently $w_i$, $w_f$, and $w_{if}$ that lie within a limited domain. The domain of $(w_{if},w_i,w_f)$ is

$$w_{if}\geqslant 1, \quad 2w_{if}w_iw_f-w_{if}^2-w_i^2-w_f^2+1\geqslant 0, \tag{10}$$

which implies

$$w_i\geqslant 1, \quad w_f\geqslant 1, \tag{11}$$

and is equivalent to

$$w_i\geqslant 1, \quad w_f\geqslant 1,$$
$$w_iw_f-\sqrt{(w_i^2-1)(w_f^2-1)} \leqslant w_{if}\leqslant w_iw_f+\sqrt{(w_i^2-1)(w_f^2-1)}. \tag{12}$$

There is a subdomain for $w_i=w_f=w$, namely,

$$w\geqslant 1, \quad 1\leqslant w_{if}\leqslant 2w^2-1. \tag{13}$$

Within this domain one can differentiate relative to any of these variables:

$$\frac{\partial^{p+q+r}L}{\partial w_{if}^p\partial w_i^q\partial w_f^r}=\frac{\partial^{p+q+r}R}{\partial w_{if}^p\partial w_i^q\partial w_f^r}, \tag{14}$$

and obtain different sum rules taking different limits to the frontier of the domain, e.g.,

$$w_{if}\to 1, \quad w_i=w_f=w,$$

$$\text{or}\quad w_i \to 1, \quad w_{if} = w_f = w,$$

$$\text{or}\quad w_f \to 1, \quad w_{if} = w_i = w. \tag{15}$$

A last general remark. In the SRs we will consider the sum over discrete intermediate ground state or excited $D$ mesons. However, our results have a wider generality, as they can include a possible continuum. Such a continuum would be only a slight technical complication, as it can also be expanded into $j^P$ states, and the sum over discrete states would become an integral, without any conceptual change in the final results.

The paper is organized as follows. In Sec. II we write down the general form of the SRs in the heavy quark limit for a general pair of currents $\overline{h}_{v'}^{(c)}\Gamma_1 h_{v_i}^{(b)}$, $\overline{h}_{v_f}^{(b)}\Gamma_2 h_{v'}^{(c)}$, making explicit the intermediate states $\frac{1}{2}^-$, $\frac{1}{2}^+$, $\frac{3}{2}^+$, and $\frac{3}{2}^-$ as well, in order to have control on high powers of the recoil $(w-1)$. In Sec. III we derive the sum rules (in particular the Bjorken and Uraltsev SRs) for the axial currents $\{\Gamma_1,\Gamma_2\} = \{\not{v}_i\gamma_5, \not{v}_f\gamma_5\}$, and in Sec. IV similarly for the vector currents $\{\Gamma_1,\Gamma_2\} = \{\not{v}_i, \not{v}_f\}$. In Sec. V we underline a new class of sum rules with implications, in particular, for the slope and curvature of $\xi(w)$. Moreover, we demonstrate that higher excited states give a vanishing contribution to these SRs. In Sec. VI we write down a lower bound on the curvature of $\xi(w)$ and in Sec. VII we point out some phenomenological remarks in connection with the Bakamjian-Thomas class of relativistic quark models. In Sec. VIII we conclude. In Appendix A we construct the general formula for the projector on the polarization tensors of particles of arbitrary spin. With it, we deduce a formula that is needed in the calculation of the contributions to the sum rules of higher excited states. Using this general result, we have recently obtained rigorous bounds on all derivatives of the IW function $\xi(w)$ [15]. For the curvature $\sigma^2 = \xi''(1)$ we find in the present paper the same bound using a different method and making a sensible phenomenological hypothesis. Finally, in Appendix B we give a derivation of Bjorken and Uraltsev SRs with the currents $\{\Gamma_1,\Gamma_2\} = \{\not{v}_i, \not{v}_i\}$ and initial and final states $B^{*(\lambda_i)}(v_i)$, $B^{*(\lambda_f)}(v_f)$, a manifestly covariant version of those states and currents used by Uraltsev, $\{\Gamma_1,\Gamma_2\} = \{\gamma^0,\gamma^0\}$ in the rest frame of the initial $B^{*(\lambda_i)}(1,\mathbf{0})$. Of course, this choice of the vector current would make simpler the calculation of radiative corrections to the sum rules than in the case, say, of the axial current. But radiative corrections are outside the scope of the present paper, which adopts the strict heavy quark limit.

## II. GENERAL FORM OF THE SUM RULES IN THE HEAVY QUARK LIMIT

The RHS is written, in the heavy quark limit, since then $v'_c = v'$,

$$R(w_{if}, w_i, w_f) = -2\xi(w_{if})\mathrm{Tr}[\overline{\mathcal{B}}_f(v_f)\Gamma_2 P'_+ \Gamma_1 \mathcal{B}_i(v_i)]. \tag{16}$$

Let us decompose the LHS into contributions of the different intermediate states: as intermediate states, we will consider the $0^-_{1/2}$, $1^-_{1/2}$, and the orbitally excited states $2^+_{3/2}$, $1^+_{3/2}$, $0^+_{1/2}$, $1^+_{1/2}$ (with the tower of their radial excitations). Moreover, to have some control of the SRs near zero recoil, it is important to have an idea of the contributions of higher orbital excitations. To this purpose, we will take into account the $\frac{3}{2}^-$ intermediate states, namely, the states $2^-_{3/2}$ and $1^-_{3/2}$.

Let us now write down the $4\times 4$ matrices of the lower $j^P$ states [4,14]. The matrices for the $\frac{1}{2}^-$ mesons read

$$0^-_{1/2}: \quad \mathcal{M}(v) = P_+(-\gamma_5),$$

$$1^-_{1/2}: \quad \mathcal{M}(v) = P_+ \varepsilon_v^\mu \gamma_\mu, \tag{17}$$

where $P_+$ is the projector

$$P_+ = \frac{1+\not{v}}{2}. \tag{18}$$

The $4\times 4$ matrices of the $\frac{3}{2}^+$ states are given by the four-vectors

$$2^+_{3/2}: \quad \mathcal{M}^\mu(v) = P_+ \varepsilon_v^{\mu\nu}\gamma_\mu,$$

$$1^+_{3/2}: \quad \mathcal{M}^\mu(v) = -\sqrt{3/2}P_+ \varepsilon_v^\nu \gamma_5 \left[ g_\nu^\mu - \frac{1}{3}\gamma_\nu(\gamma^\mu - v^\mu)\right], \tag{19}$$

and those of the $\frac{1}{2}^+$ states are given by [4]

$$0^+_{1/2}: \quad \mathcal{M}(v) = P_+,$$

$$1^+_{1/2}: \quad \mathcal{M}(v) = P_+ \varepsilon_v^\mu \gamma_5 \gamma_\mu. \tag{20}$$

Finally, those of the $\frac{3}{2}^-$ states will be obtained from Eq. (19) by multiplying on the right by $(-\gamma_5)$:

$$2^-_{3/2}: \quad \mathcal{M}^\mu(v) = P_+ \varepsilon_v^{\mu\nu}\gamma_\nu(-\gamma_5),$$

$$1^-_{3/2}: \quad \mathcal{M}^\mu(v) = \sqrt{3/2}P_+ \varepsilon_v^\nu \left[ g_\nu^\mu - \frac{1}{3}\gamma_\nu(\gamma^\mu + v^\mu)\right]. \tag{21}$$

The corresponding matrix elements, for a current given by the Dirac matrix $\Gamma$, read [4]

$$\langle D^{(n)}(\tfrac{1}{2}^-)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|B(\tfrac{1}{2}^-)(v)\rangle = \xi^{(n)}(w)\mathrm{Tr}[\overline{\mathcal{D}}(v')\Gamma\mathcal{B}(v)], \tag{22}$$

$$\langle D^{(n)}(\tfrac{3}{2}^+)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|B(\tfrac{1}{2}^-)(v)\rangle = \sqrt{3}\,\tau_{3/2}^{(n)}(w)\mathrm{Tr}[v_\mu\overline{\mathcal{D}}^\mu(v')\Gamma\mathcal{B}(v)], \tag{23}$$

$$\langle D^{(n)}(\tfrac{1}{2}^+)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|B(\tfrac{1}{2}^-)(v)\rangle = 2\,\tau_{1/2}^{(n)}(w)\mathrm{Tr}[\overline{\mathcal{D}}(v')\Gamma\mathcal{B}(v)], \tag{24}$$

$$\langle D^{(n)}(\tfrac{3}{2}^-)(v')|\overline{h}_{v'}^{(c)}\Gamma h_v^{(b)}|B(\tfrac{1}{2}^-)(v)\rangle = \sqrt{3}\,\sigma_{3/2}^{(n)}(w)\mathrm{Tr}[v_\mu\overline{\mathcal{D}}^\mu(v')\Gamma\mathcal{B}(v)], \tag{25}$$

where $w=v\cdot v'$, $n$ is a radial quantum number, and, in analogy with $\tau_{3/2}(w)$, we have called $\sigma_{3/2}(w)$ the IW function between the ground state and the $\frac{3}{2}^-$ states. As pointed out in [4], $\sigma_{3/2}(w)$ need not vanish at $w=1$, since the current matrix elements vanish in the heavy quark limit. The notation $\xi^{(n)}(w)$, $\tau_{1/2}^{(n)}(w)$, and $\tau_{3/2}^{(n)}(w)$ is that of Isgur and Wise [1].

In what follows, we set the different IW functions to be real.

The contributions of the $0^-_{1/2}$, $1^-_{1/2}$ states are written as

$$L(0^-_{1/2})=\mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\overline{\Gamma}_2 P'_+(-\gamma_5)]\mathrm{Tr}[\gamma_5 P'_+\Gamma_1\mathcal{B}_i(v_i)]\times\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f), \tag{26}$$

$$L(1^-_{1/2})=\sum_\lambda \varepsilon'^{(\lambda)\mu}\varepsilon'^{(\lambda)*\nu}\mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\overline{\Gamma}_2 P'_+\gamma_\nu]\times\mathrm{Tr}[\gamma_\mu P'_+\Gamma_1\mathcal{B}_i(v_i)]\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f). \tag{27}$$

The contribution of the parity$+$excited states $2^+_{3/2}$, $1^+_{3/2}$, $0^+_{1/2}$, $1^+_{1/2}$ is given by the following expressions:

$$L(2^+_{3/2})=\sum_\lambda \varepsilon'^{(\lambda)\mu\nu}\varepsilon'^{(\lambda)*\rho\sigma}\mathrm{Tr}[v_{f\rho}\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+\gamma_\sigma]\times\mathrm{Tr}[v_{i\mu}\gamma_\nu P'_+\Gamma_1\mathcal{B}_i]3\sum_n \tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f), \tag{28}$$

$$L(1^+_{3/2})=\sum_\lambda \varepsilon'^{(\lambda)\nu}\varepsilon'^{(\lambda)*\sigma}\frac{3}{2}\mathrm{Tr}\left\{\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+\gamma_5\times\left[v_{f\sigma}-\frac{1}{3}\gamma_\sigma(\not{v}_f-w_f)\right]\right\}\mathrm{Tr}\left\{\left[v_{i\nu}-\frac{1}{3}(\not{v}_i-w_i)\gamma_\nu\right]\times(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i\right\}3\sum_n \tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f), \tag{29}$$

$$L(0^+_{1/2})=\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]4\times\sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f), \tag{30}$$

$$L(1^+_{1/2})=\sum_\lambda \varepsilon'^{(\lambda)\mu}\varepsilon'^{(\lambda)*\nu}\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+\gamma_5\gamma_\nu]\times\mathrm{Tr}[\gamma_\mu(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]4\sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f), \tag{31}$$

$$L(2^-_{3/2})=\sum_\lambda \varepsilon'^{(\lambda)\mu\nu}\varepsilon'^{(\lambda)*\rho\sigma}\mathrm{Tr}[v_{f\rho}\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+\gamma_\sigma(-\gamma_5)]\times\mathrm{Tr}[v_{i\mu}\gamma_5\gamma_\nu P'_+\Gamma_1\mathcal{B}_i]3\sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f), \tag{32}$$

$$L(1^-_{3/2})=\sum_\lambda \varepsilon'^{(\lambda)\nu}\varepsilon'^{(\lambda)*\sigma}\frac{3}{2}\mathrm{Tr}\left\{\overline{\mathcal{B}_f}\overline{\Gamma}_2 P'_+\left[v_{f\sigma}-\frac{1}{3}\gamma_\sigma\times(\not{v}_f+w_f)\right]\right\}\mathrm{Tr}\left\{\left[v_{i\nu}-\frac{1}{3}(\not{v}_i+w_i)\gamma_\nu\right]P'_+\Gamma_1\mathcal{B}_i\right\}\times 3\sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f). \tag{33}$$

It is convenient to introduce the tensors

$$T^{\mu\nu}=\sum_\lambda \varepsilon'^{(\lambda)\mu}\varepsilon'^{(\lambda)*\nu}, \tag{34}$$

$$T^{\mu\nu,\rho\sigma}=\sum_\lambda \varepsilon'^{(\lambda)\mu\nu}\varepsilon'^{(\lambda)*\rho\sigma}. \tag{35}$$

The polarizations of the vector and tensor intermediate states of velocity $v'$ satisfy $\varepsilon'^{(\lambda)}\cdot v'=\varepsilon'^{(\lambda)\mu\nu}v'_\nu=0$. Moreover, the polarization tensor $\varepsilon'^{(\lambda)\mu\nu}$ is symmetric in $(\mu\nu)$ and traceless, $\varepsilon'^{(\lambda)\mu}_{\mu}=0$. One can show that these tensors can be written

$$T^{\mu\nu}=-g^{\mu\nu}+v'^{\mu}v'^{\nu}, \tag{36}$$

$$T^{\mu\nu,\rho\sigma}=\frac{1}{6}\{-2g^{\rho\sigma}g^{\mu\nu}+3[g^{\rho\mu}g^{\sigma\nu}+g^{\rho\nu}g^{\sigma\mu}]+2[g^{\rho\sigma}v'^{\mu}v'^{\nu}+v'^{\rho}v'^{\sigma}g^{\mu\nu}]-3[g^{\rho\mu}v'^{\sigma}v'^{\nu}+g^{\sigma\nu}v'^{\rho}v'^{\mu}+g^{\rho\nu}v'^{\sigma}v'^{\mu}+g^{\sigma\mu}v'^{\rho}v'^{\nu}]+4v'^{\mu}v'^{\nu}v'^{\rho}v'^{\sigma}\} \tag{37}$$

and have the following properties: $T^{\mu\nu}$ is symmetric and $T^{\mu}_{\mu}=-3$ while $T^{\mu\nu,\rho\sigma}$ is symmetric in the exchanges $(\mu\nu\leftrightarrow\rho\sigma)$, $(\mu\leftrightarrow\nu)$, and $(\rho\leftrightarrow\sigma)$ and satisfies $T^{\mu\nu,}{}_{\mu\nu}=+5$ (the $+$ sign comes from the fact that the polarization of a spin 2 particle can be seen as a symmetric combination of the polarizations of two spin 1 particles). With these expressions for the polarization tensors one can make more explicit the contributions of the intermediate states in the LHS of the SR (4). After some algebra one gets, from Eqs. (26)–(33), for arbitrary Dirac matrices $\Gamma_1$ and $\Gamma_2$, the following SR:

$$
\begin{aligned}
&\{[\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+(-\gamma_5)]\mathrm{Tr}[\gamma_5P'_+\Gamma_1\mathcal{B}_i]]+[-\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_\mu]\mathrm{Tr}[\gamma^\mu P'_+\Gamma_1\mathcal{B}_i]+\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]]\}\\
&\times\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f)+\frac{1}{2}\{[3(w_{if}-w_fw_i)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_\sigma]\mathrm{Tr}[\gamma^\sigma P'_+\Gamma_1\mathcal{B}_i]+(-2-2w_i-2w_f-3w_{if}+4w_iw_f)\\
&\times\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]+3\,\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\not{v}_i]\mathrm{Tr}[\not{v}_fP'_+\Gamma_1\mathcal{B}_i]-3w_i\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[\not{v}_fP'_+\Gamma_1\mathcal{B}_i]\\
&-3w_f\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\not{v}_i]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]]+[-(1+w_i)(1+w_f)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5\gamma_\sigma]\mathrm{Tr}[\gamma^\sigma(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]\\
&+(1-9w_{if}+4w_iw_f-2w_i-2w_f)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5]\mathrm{Tr}[(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]-3(1+w_f)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5\not{v}_i]\\
&\times\mathrm{Tr}[(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]-3(1+w_i)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5]\mathrm{Tr}[\not{v}_f(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]]\}\sum_n \tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f)\\
&+4\{[\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]]+[-\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5\gamma_\sigma]\mathrm{Tr}[\gamma^\sigma(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]+\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_5]\\
&\times\mathrm{Tr}[(-\gamma_5)P'_+\Gamma_1\mathcal{B}_i]]\}\sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f)+\frac{1}{2}\{[3(w_{if}-w_fw_i)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_\sigma(-\gamma_5)]\mathrm{Tr}[\gamma_5\gamma^\sigma P'_+\Gamma_1\mathcal{B}_i]\\
&+(-2+2w_i+2w_f-3w_{if}+4w_iw_f)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+(-\gamma_5)]\mathrm{Tr}[\gamma_5P'_+\Gamma_1\mathcal{B}_i]+3\,\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\not{v}_i(-\gamma_5)]\mathrm{Tr}[\gamma_5\not{v}_fP'_+\Gamma_1\mathcal{B}_i]\\
&-3w_i\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+(-\gamma_5)]\mathrm{Tr}[\gamma_5\not{v}_fP'_+\Gamma_1\mathcal{B}_i]-3w_f\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\not{v}_i(-\gamma_5)]\mathrm{Tr}[\gamma_5P'_+\Gamma_1\mathcal{B}_i]]+[-(w_i-1)(w_f-1)\\
&\times\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\gamma_\sigma]\mathrm{Tr}[\gamma^\sigma P'_+\Gamma_1\mathcal{B}_i]+(1-9w_{if}+4w_iw_f+2w_i+2w_f)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]+3(w_f-1)\\
&\times\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+\not{v}_i]\mathrm{Tr}[P'_+\Gamma_1\mathcal{B}_i]+3(w_i-1)\mathrm{Tr}[\overline{\mathcal{B}_f}\overline{\Gamma}_2P'_+]\mathrm{Tr}[\not{v}_fP'_+\Gamma_1\mathcal{B}_i]]\}\sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f)\\
&+\text{contribution from other excited states}=-2\xi(w_{if})\mathrm{Tr}[\overline{\mathcal{B}_f}(v_f)\overline{\Gamma}_2P'_+\Gamma_1\mathcal{B}_i(v_i)].
\end{aligned}
\tag{38}
$$

In the RHS the function $\xi(w_{if})$ must match the corresponding function of $w_{if}$ that one would get by summing over all possible intermediate states. In this formula, the coefficient of $\Sigma_n\xi^{(n)}(w_i)\xi^{(n)}(w_f)$ is the contribution of the $0^-_{1/2}$ (first bracket) and the $1^-_{1/2}$ (second bracket) states. Likewise, the coefficient of $\Sigma_n\tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f)$ is the contribution of the $2^+_{3/2}$ (first bracket) and the $1^+_{3/2}$ (second bracket) states. The coefficient of $\Sigma_n\tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f)$ is the contribution of the $0^+_{1/2}$ (first bracket) and the $1^+_{1/2}$ (second bracket) states. Finally, the coefficient of $\Sigma_n\sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f)$ is the contribution of the $2^-_{3/2}$ (first bracket) and the $1^-_{3/2}$ (second bracket) states.

What we called $L(w_{if},w_i,w_f)$ and $R(w_{if},w_i,w_f)$ in Sec. I are given now explicitly by Eq. (38). We will now consider the sum rules given by Eq. (14). However, since we have included only a limited number of intermediate states, it would be dangerous to draw conclusions from sum rules for $p,q,r\geqslant 2$, because missing intermediate states could contribute to the desired order. Therefore, we will limit ourselves to $(p,q,r)=(0,0,0),(1,0,0),(0,1,0),(0,0,1)$. The consideration of the states $\frac{3}{2}^-$ will give us some control over higher powers of $(w-1)$. In the main text, we will limit ourselves to currents that give functions $L(w_{if},w_i,w_f)$ and $R(w_{if},w_i,w_f)$ symmetric in $w_i,w_f$. We are then limited to the following relations from the different derivatives and boundary conditions:

$$L(w_{if},w_i,w_f)|_{w_{if}=1,w_i=w_f=w}=R(w_{if},w_i,w_f)|_{w_{if}=1,w_i=w_f=w}, \tag{39}$$

$$L(w_{if},w_i,w_f)|_{w_i=1,w_{if}=w_f=w}=R(w_{if},w_i,w_f)|_{w_i=1,w_{if}=w_f=w}, \tag{40}$$

$$\left.\frac{\partial L}{\partial w_{if}}\right|_{w_{if}=1,w_i=w_f=w}=\left.\frac{\partial R}{\partial w_{if}}\right|_{w_{if}=1,w_i=w_f=w}, \tag{41}$$

$$\left.\frac{\partial L}{\partial w_{if}}\right|_{w_i=1,w_{if}=w_f=w}=\left.\frac{\partial R}{\partial w_{if}}\right|_{w_i=1,w_{if}=w_f=w}, \tag{42}$$

$$\left.\frac{\partial L}{\partial w_i}\right|_{w_i=1,w_{if}=w_f=w}=\left.\frac{\partial R}{\partial w_i}\right|_{w_i=1,w_{if}=w_f=w}, \tag{43}$$

$$\left.\frac{\partial L}{\partial w_i}\right|_{w_f=1,w_i=w_{if}=w}=\left.\frac{\partial R}{\partial w_i}\right|_{w_f=1,w_i=w_{if}=w}, \tag{44}$$

$$\left.\frac{\partial L}{\partial w_i}\right|_{w_{if}=1, w_i=w_f=w} = \left.\frac{\partial R}{\partial w_i}\right|_{w_{if}=1, w_i=w_f=w}. \tag{45}$$

In Appendix B, we consider a manifestly covariant version of the Uraltsev case, where the functions $L(w_{if}, w_i, w_f)$ and $R(w_{if}, w_i, w_f)$ are not symmetric in $w_i, w_f$. In our conclusion we discuss the perspectives and outlook for these nonsymmetric cases.

## III. THE AXIAL CURRENT: A SIMPLE COVARIANT DERIVATION OF BJORKEN AND URALTSEV SUM RULES

To illustrate the method, let us now particularize to the simple case

$$\mathcal{B}_i = P_{i+}(-\gamma_5), \quad \mathcal{B}_f = P_{f+}(-\gamma_5),$$
$$\Gamma_1 = \not{v}_i \gamma_5, \quad \Gamma_2 = \not{v}_f \gamma_5, \tag{46}$$

where the currents are projected along the initial and final velocities.

In this symmetric situation between currents and initial and final states, a number of intermediate states do not contribute, and the calculation simplifies considerably. One has, namely,

$$L(0^-_{1/2}) = L(1^+_{3/2}) = L(1^+_{1/2}) = L(2^-_{3/2}) = 0, \tag{47}$$

and the SR (38) is written

$$(w_i w_f - w_{if}) \sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f) + [3(w_i w_f - w_{if})^2 - (w_i^2-1)(w_f^2-1)] \sum_n \tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f) + 4(w_i-1)(w_f-1)$$
$$\times \sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f) + 2(w_i-1)(w_f-1)(w_i w_f - w_{if}) \sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f) + \text{contribution from other excited states}$$
$$= -(1 - w_i - w_f + w_{if})\xi(w_{if}). \tag{48}$$

The symmetry of Eq. (48) in $(w_i, w_f)$ follows from the symmetric choice (46) of currents and states.

We assume now that the higher state contributions are, at most, of the same order in $(w-1)$ as the $\frac{3}{2}^-$ states, that are included in the calculation. This conjecture will be demonstrated in Sec. V. Equations (40), (42), and (44) are trivial [giving $0=0$ or $\xi(w)=\xi(w)$], while Eqs. (39), (41), (43), and (45) give, respectively (the contribution of higher excited states is denoted by $+\cdots$),

$$(w^2-1)\sum_n [\xi^{(n)}(w)]^2 + 2(w^2-1)^2 \sum_n [\tau_{3/2}^{(n)}(w)]^2 + 4(w-1)^2 \sum_n [\tau_{3/2}^{(n)}(w)]^2 + 2(w+1)(w-1)^3 \sum_n [\sigma_{3/2}^{(n)}(w)]^2 + \cdots$$
$$= 2(w-1), \tag{49}$$

$$-\sum_n [\xi^{(n)}(w)]^2 - 6(w^2-1)\sum_n [\tau_{3/2}^{(n)}(w)]^2 - 2(w-1)^2 \sum_n [\sigma_{3/2}^{(n)}(w)]^2 + \cdots = -1 - 2\rho^2(w-1), \tag{50}$$

$$2(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w) - 4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w) + \cdots = \xi(w), \tag{51}$$

$$w\sum_n [\xi^{(n)}(w)]^2 + (w^2-1)\sum_n \xi^{(n)}(w)\xi^{(n)\prime}(w) + 2(w^2-1)\left\{2w\sum_n [\tau_{3/2}^{(n)}(w)]^2 + (w^2-1)\sum_n \tau_{3/2}^{(n)}(w)\tau_{3/2}^{(n)\prime}(w)\right\} + 4(w-1)$$
$$\times\left\{\sum_n [\tau_{1/2}^{(n)}(w)]^2 + (w-1)\sum_n \tau_{1/2}^{(n)}(w)\tau_{1/2}^{(n)\prime}(w)\right\} + 2(w-1)^2\left\{(2w+1)\sum_n [\sigma_{3/2}^{(n)}(w)]^2 + (w^2-1)\sum_n \sigma_{3/2}^{(n)}(w)\sigma_{3/2}^{(n)\prime}(w)\right\}$$
$$+\cdots = 1. \tag{52}$$

Dividing Eq. (49) by $2(w-1)$ one gets the Bjorken SR [1]:

$$\frac{w+1}{2}\sum_n [\xi^{(n)}(w)]^2+(w-1)\left\{2\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w+1)^2 \times\sum_n [\tau_{3/2}^{(n)}(w)]^2\right\}+(w+1)(w-1)^2 \times\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=1, \tag{53}$$

where the $\frac{3}{2}^-$ states have been included explicitly.

Equation (50) gives, at order $(w-1)$,

$$1-2\rho^2(w-1)+12\sum_n [\tau_{3/2}^{(n)}(1)]^2(w-1)=1+2\rho^2(w-1), \tag{54}$$

implying

$$\rho^2=3\sum_n [\tau_{3/2}^{(n)}(1)]^2, \tag{55}$$

which, combined with the first order in $(w-1)$ of the Bjorken SR (53),

$$\rho^2=\frac{1}{4}+\sum_n [\tau_{1/2}^{(n)}(1)]^2+2\sum_n [\tau_{3/2}^{(n)}(1)]^2 \tag{56}$$

gives the Uraltsev SR [6]

$$\sum_n [\tau_{3/2}^{(n)}(1)]^2-\sum_n [\tau_{1/2}^{(n)}(1)]^2=\frac{1}{4}. \tag{57}$$

Equation (51) yields also the Uraltsev SR for $w=1$. Notice the important point that in this equation the contribution of the IW functions $\sigma_{3/2}^{(n)}(w)$ *vanishes identically*.

Finally, Eq. (52) at $O[(w-1)]$ gives again the Bjorken SR under the form (56).

## IV. THE CASE OF THE VECTOR CURRENT

Let us now consider the vector current, i.e.,

$$\mathcal{B}_i=P_{i+}(-\gamma_5), \quad \mathcal{B}_f=P_{f+}(-\gamma_5),$$

$$\Gamma_1=\not{v}_i, \quad \Gamma_2=\not{v}_f. \tag{58}$$

In this particular case, a number of different intermediate states do not contribute, namely,

$$L(1^-_{1/2})=L(2^+_{3/2})=L(0^+_{1/2})=L(1^-_{3/2})=0, \tag{59}$$

and the SR (38) is written

$$(w_i+1)(w_f+1)\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f)+2(w_i+1)(w_f+1)(w_iw_f-w_{if})\sum_n \tau_{3/2}^{(n)}(w_i)\tau_{3/2}^{(n)}(w_f)+4(w_iw_f-w_{if}) \times\sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f)+[3(w_iw_f-w_{if})^2-(w_i^2-1)(w_f^2-1)]\sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f) +\text{contribution from other excited states}=(w_{if}+1+w_f+w_i)\xi(w_{if}), \tag{60}$$

where the first, second, third, and fourth terms in the RHS come from the states $0^-_{1/2}$, $1^+_{3/2}$, $1^+_{1/2}$, and $2^-_{3/2}$, respectively.

Equation (40) is trivial $[\xi(w)=\xi(w)]$, while Eqs. (39), (41)–(45) give now, respectively,

$$(w+1)^2\sum_n [\xi^{(n)}(w)]^2+2(w^2-1)\left\{(w+1)^2\sum_n [\tau_{3/2}^{(n)}(w)]^2+2\sum_n [\tau_{1/2}^{(n)}(w)]^2\right\}+2(w^2-1)^2\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=2(w+1), \tag{61}$$

$$-2(w+1)^2\sum_n [\tau_{3/2}^{(n)}(w)]^2-4\sum_n [\tau_{1/2}^{(n)}(w)]^2-6(w^2-1)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=1-2(w+1)\rho^2 \tag{62}$$

$$-4(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)-4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w)+\cdots=\xi(w)+2(w+1)\xi'(w), \tag{63}$$

$$w\xi(w)+2(w+1)\sum_n \xi^{(n)\prime}(1)\xi^{(n)}(w)+4w(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)+4w\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w) -2(w^2-1)\sum_n \sigma_{3/2}^{(n)}(1)\sigma_{3/2}^{(n)}(w)+\cdots=0, \tag{64}$$

$$\xi(w)+2(w+1)\xi'(w)+4(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)+4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w)+\cdots=0, \tag{65}$$

$$\begin{aligned}&(w+1)\sum_n [\xi^{(n)}(w)]^2+(w+1)^2\sum_n \xi^{(n)}(w)\xi^{(n)\prime}(w)+2(w+1)^2\left\{(2w-1)\sum_n [\tau_{3/2}^{(n)}(w)]^2+(w^2-1)\sum_n \tau_{3/2}^{(n)}(w)\tau_{3/2}^{(n)\prime}(w)\right\}\\&+4\left\{w\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w^2-1)\sum_n \tau_{1/2}^{(n)}(w)\tau_{1/2}^{(n)\prime}(w)\right\}\\&+(w^2-1)\left\{4w\sum_n [\sigma_{3/2}^{(n)}(w)]^2+2(w^2-1)\sum_n \sigma_{3/2}^{(n)}(w)\sigma_{3/2}^{(n)\prime}(w)\right\}+\cdots=1.\end{aligned} \tag{66}$$

Notice an important point, namely, that in Eq. (63), identical to Eq. (65), the contribution of the IW functions $\sigma_{3/2}^{(n)}(w)$ *vanishes identically*.

Dividing Eq. (61) by $2(w+1)$ one gets the Bjorken SR for all $w$ [Eq. (53)]. Equations (62)–(66) imply, for $w=1$, the Bjorken SR (56) for the elastic slope $\rho^2$.

## V. A NEW CLASS OF SUM RULES AND THE CONTRIBUTION OF HIGHER EXCITED STATES

Among the SRs that we obtained in Secs. III and IV, there is a new class that involves the IW functions $\xi^{(n)}(w)$, $\tau_{3/2}^{(n)}(w)$, $\tau_{1/2}^{(n)}(w)$,... for any $w$, and at zero recoil $w=1$. The relation that we got from the axial vector currents is

$$2(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)-4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w)+\cdots = \xi(w) \tag{67}$$

while we obtained, from the vector current,

$$-4(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)-4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w)+\cdots = \xi(w)+2(w+1)\xi'(w), \tag{68}$$

$$\begin{aligned}&w\xi(w)+2(w+1)\sum_n \xi^{(n)\prime}(1)\xi^{(n)}(w)+4w(w+1)\\&\times\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w)+4w\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w)\\&-2(w^2-1)\sum_n \sigma_{3/2}^{(n)}(1)\sigma_{3/2}^{(n)}(w)+\cdots=0.\end{aligned} \tag{69}$$

The first equation (67) is a generalization of the Uraltsev SR for $w\neq 1$, which reduces to Eq. (57) for $w=1$, while the other two (68) and (69) give, taking $w=1$, the Bjorken SR (56) for the slope $\rho^2$.

Let us concentrate on Eqs. (67) and (68). An important feature of these relations is that the contribution from the $\frac{3}{2}^-$ states vanishes identically. This is not the case, however, for relation (69).

We will now give a proof that no other higher intermediate states contribute to the sum rules (67) and (68).

Following the work of Falk [16], we write first the $4\times 4$ matrices of the whole tower of $j^P$ states, generalizing the notation we have given above (17)–(21), where $k=j-\frac{1}{2}$, $J$ is the spin of the state, and $\ell$ is the orbital angular momentum:

$$j=\ell+\tfrac{1}{2},\quad J=j+\tfrac{1}{2}:$$

$$\mathcal{M}^{\mu_1\cdots\mu_k}(v)=P_+\varepsilon_v^{\mu_1\cdots\mu_{k+1}}\gamma_{\mu_{k+1}}; \tag{70}$$

$$j=\ell+\tfrac{1}{2},\quad J=j-\tfrac{1}{2}:$$

$$\begin{aligned}\mathcal{M}^{\mu_1\cdots\mu_k}(v)=&-\sqrt{(2k+1)/(k+1)}P_+\gamma_5\varepsilon_v^{\nu_1\cdots\nu_k}\\&\times\left[g_{\nu_1}^{\mu_1}\cdots g_{\nu_k}^{\mu_k}-\frac{1}{2k+1}\gamma_{\nu_1}(\gamma^{\mu_1}-v^{\mu_1})\right.\\&\times g_{\nu_2}^{\mu_2}\cdots g_{\nu_k}^{\mu_k}-\cdots-\frac{1}{2k+1}\\&\left.\times g_{\nu_1}^{\mu_1}\cdots g_{\nu_{k-1}}^{\mu_{k-1}}\gamma_{\nu_k}(\gamma^{\mu_k}-v^{\mu_k})\right];\end{aligned} \tag{71}$$

$$j=\ell-\tfrac{1}{2},\quad J=j+\tfrac{1}{2}:$$

$$\mathcal{M}^{\mu_1\cdots\mu_k}(v)=P_+\varepsilon_v^{\mu_1\cdots\mu_{k+1}}\gamma_5\gamma_{\mu_{k+1}}; \tag{72}$$

$$j=\ell-\tfrac{1}{2},\quad J=j-\tfrac{1}{2}:$$

$$\begin{aligned}\mathcal{M}^{\mu_1\cdots\mu_k}(v)=&\sqrt{(2k+1)/(k+1)}P_+\varepsilon^{\nu_1\cdots\nu_k}\\&\times\left[g_{\nu_1}^{\mu_1}\cdots g_{\nu_k}^{\mu_k}-\frac{1}{2k+1}\gamma_{\nu_1}(\gamma^{\mu_1}+v^{\mu_1})\right.\\&\times g_{\nu_2}^{\mu_2}\cdots g_{\nu_k}^{\mu_k}-\cdots-\frac{1}{2k+1}\\&\left.\times g_{\nu_1}^{\mu_1}\cdots g_{\nu_{k-1}}^{\mu_{k-1}}\gamma_{\nu_k}(\gamma^{\mu_k}+v^{\mu_k})\right].\end{aligned} \tag{73}$$

For a transition of the type $\mathcal{B}^{\mu_1\cdots\mu_k}(v)\rightarrow\mathcal{D}^{\nu_1\cdots\nu_{k'}}(v')$, the preceding expressions have to be contracted with the tensor containing all possible independent IW functions ($k'>k$):

$$\xi_{\nu_1\cdots\nu_{k'},\mu_1\cdots\mu_k}=(-1)^k(v-v')_{\nu_{k+1}}\cdots(v-v')_{\nu_{k'}} \times[\xi_0^{(k',k)}(w)g_{\nu_1\mu_1}\cdots g_{\nu_k\mu_k}+\xi_1^{(k',k)}(w) \times(v-v')_{\nu_1}(v-v')_{\mu_1}g_{\nu_2\mu_2}\cdots g_{\nu_k\mu_k}+\cdots +\xi_k^{(k',k)}(w)(v-v')_{\nu_1} \times(v-v')_{\mu_1}\cdots(v-v')_{\nu_k}(v-v')_{\mu_k}]. \quad (74)$$

However, we are here interested in the transitions between the ground state and the excited states $\frac{1}{2}^-\to j^P$, i.e., $k=0$, and the tensor (74) becomes, in this case,

$$\xi_{\mu_1\cdots\mu_k}=(v-v')_{\mu_1}\cdots(v-v')_{\mu_k}\xi_0^{(k)}(w). \quad (75)$$

Then the matrix elements are written, for the different cases,

$$\langle D^{(n)}_{(j=\ell+1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\Gamma h^{(b)}_v|B^{(*)}(v)\rangle =\tau^{(\ell)(n)}_{\ell+1/2}(w)v_{\mu_1}\cdots v_{\mu_k}\varepsilon'^{*\mu_1\cdots\mu_{k+1}} \times\mathrm{Tr}[\gamma_{\mu_{k+1}}P'_+\Gamma\mathcal{B}(v)], \quad (76)$$

$$\langle D^{(n)}_{(j=\ell+1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\Gamma h^{(b)}_v|B^{(*)}(v)\rangle =\sqrt{(2k+1)/(k+1)}\,\tau^{(\ell)(n)}_{\ell+1/2}(w) \times\varepsilon'^{*\nu_1\cdots\nu_k}\mathrm{Tr}\left\{\left[v_{\nu_1}\cdots v_{\nu_k}-\frac{1}{2k+1}(\not{v}-w) \times\gamma_{\nu_1}v_{\nu_2}\cdots v_{\nu_k}-\cdots-\frac{1}{2k+1}v_{\nu_1}\cdots v_{\nu_{k-1}} \times(\not{v}-w)\gamma_{\nu_k}\right]\gamma_5P'_+\Gamma\mathcal{B}(v)\right\}, \quad (77)$$

$$\langle D^{(n)}_{(j=\ell-1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\Gamma h^{(b)}_v|B^{(*)}(v)\rangle =\tau^{(\ell)(n)}_{\ell-1/2}(w)v_{\mu_1}\cdots v_{\mu_k}\varepsilon'^{*\mu_1\cdots\mu_{k+1}} \times\mathrm{Tr}[\gamma_{\mu_{k+1}}(-\gamma_5)P'_+\Gamma\mathcal{B}(v)], \quad (78)$$

$$\langle D^{(n)}_{(j=\ell-1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\Gamma h^{(b)}_v|B^{(*)}(v)\rangle =\sqrt{(2k+1)/(k+1)}\,\tau^{(\ell)(n)}_{\ell-1/2}(w) \times\varepsilon'^{*\nu_1\cdots\nu_k}\mathrm{Tr}\left\{\left[v_{\nu_1}\cdots v_{\nu_k}-\frac{1}{2k+1} \times(\not{v}+w)\gamma_{\nu_1}v_{\nu_2}\cdots v_{\nu_k}-\cdots-\frac{1}{2k+1}v_{\nu_1}\cdots v_{\nu_{k-1}} \times(\not{v}+w)\gamma_{\nu_k}\right]P'_+\Gamma\mathcal{B}(v)\right\}. \quad (79)$$

In all these relations we have made use of the orthogonality condition

$$v'_{\nu_i}\varepsilon'^{*\nu_1\cdots\nu_k}=0 \quad (i=1,\ldots,k). \quad (80)$$

$\mathcal{B}(v)$ denotes the $4\times4$ matrix of the ground state, $B$ or $B^*$ [Eq. (17)]. The functions $\tau^{(\ell)(n)}_{j=\ell\pm1/2}(w)$ are the generalizations to arbitrary $j$ of the IW functions introduced above, namely,

$$\tau^{(0)}_{1/2}(w)\equiv\xi(w), \quad \tau^{(1)}_{3/2}(w)\equiv\sqrt{3}\,\tau_{3/2}(w),$$

$$\tau^{(1)}_{1/2}(w)\equiv2\,\tau_{1/2}(w), \quad \tau^{(2)}_{3/2}(w)\equiv\sqrt{3}\,\sigma_{3/2}(w), \quad (81)$$

with an implicit radial quantum number $n$. Therefore, $\tau^{(1)}_{3/2}(w)$ and $\tau^{(1)}_{1/2}(w)$ are respectively identical to the functions $\tau(w)$ and $\zeta(w)$ defined by Leibovich *et al.* [4]. The superscript $\ell$ in $\tau^{(\ell)(n)}_{\ell\pm1/2}(w)$ is necessary as it indicates the parity, since for a given $j=\ell\pm\frac{1}{2}\geqslant\frac{1}{2}$, there are two possible values for $\ell=j\pm\frac{1}{2}$, and therefore two possible parities $P=(-1)^{\ell+1}$.

Considering now the $B$ meson, as in the preceding sections,

$$\mathcal{B}(v)=P_+(-\gamma_5), \quad (82)$$

we compute the different matrix elements. Remembering that $k=j-\frac{1}{2}$ one obtains the following results.

Vector current:

$$\langle D^{(n)}_{(j=\ell+1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}h^{(b)}_v|B(v)\rangle =\langle D^{(n)}_{(j=\ell-1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}h^{(b)}_v|B(v)\rangle=0, \quad (83)$$

$$\langle D^{(n)}_{(j=\ell+1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}h^{(b)}_v|B(v)\rangle =-\sqrt{(\ell+1)/(2\ell+1)}(w+1)\,\tau^{(\ell)(n)}_{\ell+1/2}(w) \times v_{\mu_1}\cdots v_{\mu_\ell}\varepsilon'^{*\mu_1\cdots\mu_\ell}, \quad (84)$$

$$\langle D^{(n)}_{(j=\ell-1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}h^{(b)}_v|B(v)\rangle =-\tau^{(\ell)(n)}_{\ell-1/2}(w)v_{\mu_1}\cdots v_{\mu_\ell}\varepsilon'^{*\mu_1\cdots\mu_\ell}. \quad (85)$$

Axial current:

$$\langle D^{(n)}_{(j=\ell+1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}\gamma_5h^{(b)}_v|B(v)\rangle =\langle D^{(n)}_{(j=\ell-1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}\gamma_5h^{(b)}_v|B^{(*)}(v)\rangle=0, \quad (86)$$

$$\langle D^{(n)}_{(j=\ell+1/2,J=j+1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}\gamma_5h^{(b)}_v|B(v)\rangle =-\tau^{(\ell)(n)}_{\ell+1/2}(w)v_{\mu_1}\cdots v_{\mu_{\ell+1}}\varepsilon'^{*\mu_1\cdots\mu_{\ell+1}}, \quad (87)$$

$$\langle D^{(n)}_{(j=\ell-1/2,J=j-1/2)}(v')|\overline{h}^{(c)}_{v'}\not{v}\gamma_5h^{(b)}_v|B(v)\rangle =-\sqrt{\ell/(2\ell-1)}(w-1)\,\tau^{(\ell)(n)}_{\ell-1/2}(w) \times v_{\mu_1}\cdots v_{\mu_{\ell-1}}\varepsilon'^{*\mu_1\cdots\mu_{\ell-1}}. \quad (88)$$

We can now write down the contributions to the LHS of the SR. We proceed as in Secs. III and IV adopting the symmetric cases (46) and (58). In an obvious notation, one finds the following results.

Vector current:

$$L_{(j=\ell+1/2,J=j+1/2)}=L_{(j=\ell-1/2,J=j-1/2)}=0, \tag{89}$$

$$L_{(j=\ell+1/2,J=j-1/2)}=\frac{\ell+1}{2\ell+1}(w_i+1)(w_f+1)\times S_\ell(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i)\tau^{(\ell)(n)}_{\ell+1/2}(w_f), \tag{90}$$

$$L_{(j=\ell+1/2,J=j+1/2)}=S_\ell(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f). \tag{91}$$

Axial current:

$$L_{(j=\ell+1/2,J=j-1/2)}=L_{(j=\ell-1/2,J=j+1/2)}=0, \tag{92}$$

$$L_{(j=\ell+1/2,J=j+1/2)}=S_{\ell+1}(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell+1/2}(w_i)\tau^{(\ell)(n)}_{\ell+1/2}(w_f), \tag{93}$$

$$L_{(j=\ell-1/2,J=j-1/2)}=\frac{\ell}{2\ell-1}(w_i-1)(w_f-1)\times S_{\ell-1}(w_i,w_f,w_{if})\sum_n \tau^{(\ell)(n)}_{\ell-1/2}(w_i)\tau^{(\ell)(n)}_{\ell-1/2}(w_f). \tag{94}$$

In all these relations, the quantity $S_n$ defined by

$$S_n=v_{i\nu_1}\cdots v_{i\nu_n}v_{f\mu_1}\cdots v_{f\mu_n}T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n},$$

where

$$T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n}=\sum_\lambda \varepsilon'^{(\lambda)*\nu_1\cdots\nu_n}\varepsilon'^{(\lambda)\mu_1\cdots\mu_n} \tag{95}$$

depends only on the four-velocity $v'$, and $\varepsilon'^{(\lambda)\mu_1\cdots\mu_n}$ is a symmetric tensor with vanishing contractions and transverse to $v'$ (see Appendix A).

It can be shown, as demonstrated below in Appendix A, that the scalar quantity

$$S_n=v_{i\nu_1}\cdots v_{i\nu_n}v_{f\mu_1}\cdots v_{f\mu_n}T^{\nu_1\cdots\nu_n,\mu_1\cdots\mu_n} \tag{96}$$

can be computed and is given by the following expression:

$$S_n=\sum_{0\leqslant k\leqslant n/2} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k(w_iw_f-w_{if})^{n-2k}, \tag{97}$$

where

$$C_{n,k}=(-1)^k\frac{(n!)^2}{(2n)!}\frac{(2n-2k)!}{k!(n-k)!(n-2k)!}. \tag{98}$$

We did find that in the SRs (51) and (63) or (65) the contribution of the states $\frac{3}{2}^-$ is identically zero. Using now the preceding general formulas, let us prove that not only does the contribution of the states $j^P=\frac{3}{2}^-$ vanish, but that this is also the case for any $j\geqslant\frac{5}{2}$. This result will imply that the SRs (67) and (68) are exact equations, i.e., we can drop out the $+\cdots$.

Let us begin Eq. (51), which was found with the axial vector current by differentiating with respect to $w_i$ and taking the limit $w_i=1$, $w_f=w_{if}=w$. Notice first that

$$S_n(w_i,w_f,w_{if})\big|_{w_i=1,w_f=w_{if}=w}=0 \quad (n\geqslant 1) \tag{99}$$

because of the orthogonality condition (80).

From Eqs. (97) and (98) we need to prove that

$$\frac{\partial}{\partial w_i}S_{n+1}(w_i,w_f,w_{if})\bigg|_{w_i=1,w_f=w_{if}=w}=0 \quad (n\geqslant 2), \tag{100}$$

$$\frac{\partial}{\partial w_i}(w_i-1)(w_f-1)S_n(w_i,w_f,w_{if})\bigg|_{w_i=1,w_f=w_{if}=w}=0 \quad (n\geqslant 1). \tag{101}$$

The second condition (101) is obviously satisfied because of the factor $(w_i-1)$ and the orthogonality condition (80).

The first condition (100) holds also, as can be seen from the explicit formula (97):

$$\frac{\partial}{\partial w_i}S_{n+1}(w_i,w_f,w_{if}) =\sum_{0\leqslant k\leqslant (n+1)/2} C_{n+1,k}(w_f^2-1)^k[2kw_i(w_i^2-1)^{k-1}\times(w_iw_f-w_{if})^{n+1-2k}+(n+1-2k)w_f(w_i^2-1)^k\times(w_iw_f-w_{if})^{n-2k}] \tag{102}$$

which vanishes for $w_i=1$, $w_f=w_{if}=w$ when $n\geqslant 2$. Notice that this expression does not vanish for $n=1$, which corresponds to the contribution of the $\frac{3}{2}^+$ states to the SRs.

Let us now consider Eq. (63), which was found with the vector current by derivation with respect to $w_{if}$, and taking the limit $w_i=1$, $w_f=w_{if}=w$, or Eq. (65) by derivation with respect to $w_i$, and taking the limit $w_f=1$, $w_i=w_{if}=w$. From Eq. (97) we need to prove

$$\frac{\partial}{\partial w_{if}}(w_i+1)(w_f+1)S_n(w_i,w_f,w_{if})\bigg|_{w_i=1,w_f=w_{if}=w}=0 \quad (n\geqslant 2), \tag{103}$$

$$\frac{\partial}{\partial w_{if}} S_{n+1}(w_i,w_f,w_{if})\Big|_{w_i=1,w_f=w_{if}=w} = 0 \quad (n \geqslant 1). \tag{104}$$

This is indeed the case, since

$$\frac{\partial}{\partial w_{if}} S_n(w_i,w_f,w_{if}) = -\sum_{0\leqslant k\leqslant n/2} C_{n,k}(w_i^2-1)^k(w_f^2-1)^k \times(n-2k)(w_i w_f - w_{if})^{n-2k-1} \tag{105}$$

vanishes for $w_i=1$, $w_f=w_{if}=w$ when $n\geqslant 2$. Notice that this quantity does not vanish for $n=1$, corresponding again to the contribution of the $\frac{3}{2}^+$ states to the SRs. The proof can be done also by derivation with respect to $w_i$, and taking the limit $w_f=1$, $w_i=w_{if}=w$.

In conclusion, we have demonstrated that in the SRs (67) and (68) there are no contributions from higher excitations.

We must make an important distinction between the different SRs that we have obtained. On the one hand, there are the SRs to which contribute the whole series of $j^P$ excitations. On the other hand, we have obtained two special SRs (67) and (68), where only a limited number of intermediate states contribute.

One can understand the truncation of the series in this latter case because the SRs correspond to the boundary condition $w_i=1$, $w_{if}=w_f=w$. Therefore, the matrix element $\langle D^{(n}(v')|\overline{h}^{(c)}_{v'}\Gamma_1 h^{(b)}_{v_i}|B(v_i)\rangle$ is computed at zero recoil, hence the finite number of terms in the expansion. As we have seen, the SR (67) obtained with the axial vector currents implies at zero recoil the Uraltsev SR (57). The corresponding SR from the vector current (68) is the Bjorken-type counterpart and indeed implies, at zero recoil, the Bjorken SR for the slope (56).

On the other hand, since all the SRs that we have obtained are exact relations, we can differentiate them relative to $w$ and, for a given derivative, taking the zero recoil limit $w=1$, the series will be truncated due to the higher powers of the recoil $(w-1)^\ell$ as $\ell$ increases. Therefore, one can expect to obtain information on higher derivatives of the elastic IW function $\xi(w)$, as we have done in Ref. [15].

## VI. A BOUND ON THE CURVATURE FROM THE NEW SUM RULES

In the preceding section we demonstrated that the SRs (67) and (68) do not have contributions from higher excited states, i.e., we can omit $+\cdots$ in these equations. This is an important result that means that *these SRs, involving only* $\xi(w)$, $\tau_{1/2}^{(n)}(w)$, *and* $\tau_{3/2}^{(n)}(w)$, *are exact relations for all* $w$, namely,

$$2(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w) - 4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w) = \xi(w), \tag{106}$$

$$-4(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w) - 4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)}(w) = \xi(w)+2(w+1)\xi'(w). \tag{107}$$

These relations are the main result of this paper.

Therefore, we can still differentiate relation (107):

$$-4\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)}(w) - 4(w+1)\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(w) - 4\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)\prime}(w) = \xi'(w)+2\xi'(w)+2(w+1)\xi''(w). \tag{108}$$

Expanding the elastic IW function $\xi(w)$ in powers of $(w-1)$,

$$\xi(w) = 1-\rho^2(w-1)+\frac{\sigma^2}{2}(w-1)^2+\cdots, \tag{109}$$

one obtains, at zero recoil,

$$-4\sum_n [\tau_{3/2}^{(n)}(1)]^2 - 8\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(1) - 4 \times\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)\prime}(1) = -3\rho^2+4\sigma^2, \tag{110}$$

and from relation (55) for $\rho^2$ one obtains

$$\sigma^2 = \frac{5}{12}\rho^2 - 2\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(1) - \sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)\prime}(1). \tag{111}$$

We can also differentiate relation (106) relative to $w$ and take the zero recoil limit:

$$2\sum_n [\tau_{3/2}^{(n)}(1)]^2 + 4\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(1) - 4\sum_n \tau_{1/2}^{(n)} \times(1)\tau_{1/2}^{(n)\prime}(1) = -\rho^2, \tag{112}$$

and from Eq. (55) we obtain

$$\rho^2 = -\frac{12}{5}\left[\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(1) - \sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)\prime}(1)\right]. \tag{113}$$

Combining relations (111) and (113) one obtains

$$\sigma^2 = -3\sum_n \tau_{3/2}^{(n)}(1)\tau_{3/2}^{(n)\prime}(1). \tag{114}$$

Equations (113) and (114) are important results of the present paper. We must insist on the fact that they are exact relations, as no other higher excited states contribute to the sums in the RHS.

Let us now discuss these formulas. If we make the plausible assumption

$$-\sum_n \tau_{1/2}^{(n)}(1)\tau_{1/2}^{(n)\prime}(1)>0, \tag{115}$$

the following inequality follows from Eqs. (113) and (114):

$$\sigma^2 \geqslant \frac{5}{4}\rho^2. \tag{116}$$

This inequality, from the lower bound $\rho^2 \geqslant \frac{3}{4}$ [6,12], implies the absolute bound

$$\sigma^2 \geqslant \frac{15}{16}. \tag{117}$$

The assumption (115) is valid if the $n=0$ state dominates the sum, and if $\tau_{1/2}^{(0)\prime}(1)<0$. This latter condition is very natural, since it concerns transitions between states of radial quantum number $n=0$, and therefore with no nodes in the wave function.

## VII. PHENOMENOLOGICAL REMARKS

In the Bakamjian-Thomas (BT) type of relativistic quark model, we have shown that Bjorken and Uraltsev SRs are satisfied [17]. Moreover, these SRs are approximately saturated by the $n=0$ states. We can add that the slopes of all three IW functions $\xi(w)$, $\tau_{3/2}^{(0)}(w)$, and $\tau_{1/2}^{(0)}(w)$ are negative [18]; namely, a good approximate parametrization of these functions is given by

$$\xi(w)=\left(\frac{2}{w+1}\right)^{2\rho^2}, \quad \tau_{3/2}^{(0)}(w)=\left(\frac{2}{w+1}\right)^{2\sigma_{3/2}^2},$$

$$\tau_{1/2}^{(0)}(w)=\left(\frac{2}{w+1}\right)^{2\sigma_{1/2}^2}. \tag{118}$$

In the spectroscopic model of Godfrey and Isgur (GI), one finds the results

$$\xi(1)=1, \quad \rho^2=1.02,$$

$$\tau_{1/2}^{(0)}(1)=0.22, \quad \sigma_{1/2}^2=0.83,$$

$$\tau_{3/2}^{(0)}(1)=0.54, \quad \sigma_{3/2}^2=1.50. \tag{119}$$

We observe that on approximating the RHS of Eq. (113) with the $n=0$ states this SR can be written

$$\rho^2=1.02=0.95+\text{contributions form } n\neq 0 \text{ excitations.} \tag{120}$$

The inequality (116) is satisfied also in the BT scheme, since, for example, in the GI spectroscopic model

$$\rho^2 \cong 1, \quad \sigma^2 \cong \frac{3}{2}, \tag{121}$$

and the inequality (116) becomes 3/2>5/4. Therefore the conjecture (115) is satisfied in the model. Notice that BT quark models satisfy Bjorken and Uraltsev SRs [17].

Although it remains to be proved, it is highly plausible that these models satisfy the whole set of SRs of QCD in the heavy quark limit, and therefore the new class (113) and (114).

Finally, from relation (114) we get the following result for the curvature, compared with the direct result (121) from the elastic IW function $\xi(w)$ (118):

$$\sigma^2 \cong 1.5=1.31+\text{contributions form } n\neq 0 \text{ excitations.} \tag{122}$$

We can conclude that there is an excellent qualitative agreement between the slope and the curvature of the elastic IW function as given directly from its calculation and as estimated from the SRs (113) and (114), if one assumes that the $n=0$ states dominate, as already has been checked using the Bjorken and Uraltsev sum rules.

## VIII. CONCLUSIONS AND OUTLOOK

In conclusion, within the OPE, we have presented a covariant method, using the trace formalism, to obtain sum rules in the heavy quark limit that relate the elastic Isgur-Wise function $\xi(w)$ to IW functions of transitions to excited states.

A main ingredient has been the introduction of the domain of the three variables $(w_i, w_f, w_{if})$, that allows a systematic way of exploring all possible SRs. In particular, we have given a simple and direct deduction of Bjorken and Uraltsev SRs, with generalizations of the latter for $w\neq 1$. The simplicity of the proof relies on the choice of the pseudoscalar $B$ meson $B(v_i)\to D^{(n)}(v')\to B(v_f)$ and of currents projected on the initial and final velocities $v_i$ and $v_f$, like $(\Gamma_1,\Gamma_2)=(\not{v}_i,\not{v}_f)$ or $(\not{v}_i\gamma_5,\not{v}_f\gamma_5)$. This simplifies the calculation enormously, since it gives vanishing contributions for half of the possible intermediate states. Notice that we obtain the same SRs (48) and (60), if we use $(\Gamma_1,\Gamma_2)=(\not{v}',\not{v}')$ or $(\mathbb{1},\mathbb{1})$ and $(\Gamma_1,\Gamma_2)=(\not{v}'\gamma_5,\not{v}'\gamma_5)$ or $(i\gamma_5,i\gamma_5)$.

Moreover, a new class of SR, involving on the one hand IW functions at zero recoil and on the other hand IW functions for any $w$ have been obtained. These SRs reduce to known results for $w=1$.

Among these new SRs, we have found two new relations that involve only the elastic IW function $\xi(w)$, and the excited $\tau_{1/2}^{(n)}(w)$ and $\tau_{3/2}^{(n)}(w)$, with *vanishing* contributions for all other IW functions between the ground and higher excited states. The vanishing of the states $\frac{3}{2}^-$ has been shown explicitly, using the corresponding wave function. We have generalized this result, demonstrating that all contributions of higher states with $j^\pm$, $j\geqslant\frac{5}{2}$, vanish identically. An important ingredient in the proof was a compact formula for the polarization tensor saturated with initial and final four-velocities.

These new SRs are therefore very strong and provide new results that relate the slope $\rho^2$ and the curvature $\sigma^2$ of $\xi(w)$ to $\tau_{1/2}^{(n)}(1),\tau_{3/2}^{(n)}(1)$ and $\tau_{1/2}^{(n)\prime}(1),\tau_{3/2}^{(n)\prime}(1)$. Modulo a very natural assumption, these SRs imply the bound $\sigma^2\geqslant\frac{5}{4}\rho^2$.

On the other hand, as a phenomenological remark, we have shown that these new SRs for $\rho^2$ and $\sigma^2$ are in good agreement with the numerical results obtained within the

Bakamjian-Thomas relativistic quark models, which satisfy Isgur-Wise scaling. In this framework, the SRs are saturated to great accuracy by the $n=0$ intermediate states.

Which are the prospects for this work? The main aim would be to obtain all possible usable SRs. By "usable" we mean SRs that involve only $\xi(w)$ and $\tau_{1/2}^{(n)}(w), \tau_{3/2}^{(n)}(w)$.

For the moment, we have concentrated mainly on the case, which appears to be simple, $B(v_i)\to B(v_f)$ with symmetric currents, projected along $v_i$ and $v_f$. One should also study the case of the transitions $B(v_i)\to B^*(v_f)$ and $B^*(v_i)\to B^*(v_f)$ and nonsymmetric currents like $(\Gamma_1,\Gamma_2)=(\not{v}_i,\not{v}_i)$, $(\not{v}_i,\not{v}')$, etc., or equivalently $(\Gamma_1,\Gamma_2)=(\gamma_\mu,\gamma_\nu)$, $(\gamma_\mu\gamma_5,\gamma_\nu\gamma_5)$,... for which in general all intermediate states contribute. We have explored a number of these nonsymmetric situations for the pseudoscalar $B$ meson and found confirmation of the results presented here.

The case of the $B^*$ is rather involved because of the polarization, mainly in the case of nonsymmetric currents, as used by Uraltsev in finding his SR. We have given in Appendix B our covariant version of his calculation.

A systematic complete study remains to be done and may be worthwhile. In particular, it would be interesting to check if the conjecture (115) on $\tau_{1/2}^{(n)}(w)$, satisfied by BT quark models, that leads from the SR obtained here to the inequality $\sigma^2\geqslant\frac{5}{4}\rho^2$, is or is not a result of heavy quark symmetry [19].

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## APPENDIX A: PROJECTOR ON THE POLARIZATION TENSORS

The polarization state of a relativistic boson is commonly described by a polarization tensor, generalizing the polarization vector of a spin 1 particle. The polarization tensors of a particle of integer spin $J$ are the tensors $\varepsilon_{\mu_1\cdots\mu_J}$ of tank $J$ which satisfy the following conditions: (1) Symmetry: $\varepsilon_{\mu_1\cdots\mu_J}=\varepsilon_{\mu_{\sigma(1)}\cdots\mu_{\sigma(J)}}$ for any permutation $\sigma$ of 1,...,$J$; (2) vanishing contractions (or tracelessness): $g^{\mu\mu'}\varepsilon_{\mu\mu'\mu_3\cdots\mu_J}=0$ (when $J\geqslant 2$); and (3) transversity: $v^\mu\varepsilon_{\mu\mu_2\cdots\mu_J}=0$, where $v$ is the four-velocity of the particle.

An orthornormal set of $2J+1$ polarization states will be described by a set $\varepsilon^{(\lambda)}_{\mu_1\cdots\mu_J}$ of polarization tensors satisfying the following normalization conditions:

$$g^{\mu_1\nu_1}\cdots g^{\mu_J\nu_J}\varepsilon^{(\lambda)}_{\mu_1\cdots\mu_J}(\varepsilon^{(\lambda')}_{\nu_1\cdots\nu_J})^*=(-1)^J\delta_{\lambda\lambda'}. \quad \text{(A1)}$$

Then, when summing over the intermediate states of a particle of integer spin $J$, one has to deal with the *projector on polarization tensors* $\Pi^{(v)}$ defined by

$$\Pi^{(v)}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}=\sum_{\lambda=-J}^{J}\varepsilon^{(\lambda)}_{\mu_1\cdots\mu_J}(\varepsilon^{(\lambda)}_{\nu_1\cdots\nu_J})^*. \quad \text{(A2)}$$

In this appendix, we intend to deduce an explicit expression for this tensor. The basic result is

$$v_f^{\mu_1}\cdots v_f^{\mu_J}\Pi^{(v)}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}v_i^{\nu_1}\cdots v_i^{\nu_J}=2^J\frac{(J!)^2}{(2J)!}(w_i^2-1)^{J/2}\times(w_f^2-1)^{J/2}P_J\left(\frac{w_iw_f-w_{if}}{\sqrt{(w_i^2-1)(w_f^2-1)}}\right), \quad \text{(A3)}$$

where $v_i$ and $v_f$ are arbitrary velocity four-vectors, $w_i, w_f, w_{if}$ are the following scalar products:

$$w_i=v\cdot v_i,\quad w_f=v\cdot v_f,\quad w_{if}=v_i\cdot v_f, \quad \text{(A4)}$$

and $P_n$ is the usual Legendre polynomial.

The matrix element (A3) is a polynomial in $w_i, w_f, w_{if}$. Using explicit expressions of $P_n$, one has the two following useful expressions of this polynomial:

$$v_f^{\mu_1}\cdots v_f^{\mu_J}\Pi^{(v)}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}v_i^{\nu_1}\cdots v_i^{\nu_J}=\sum_{0\leqslant k\leqslant J/2}C_{J,k}(w_i^2-1)^k(w_iw_f-w_{if})^{J-2k}(w_f^2-1)^k, \quad \text{(A5)}$$

$$v_f^{\mu_1}\cdots v_f^{\mu_J}\Pi^{(v)}_{\mu_1\cdots\mu_J;\nu_1\cdots\nu_J}v_i^{\nu_1}\cdots v_i^{\nu_J}=\sum_{0\leqslant k\leqslant J/2}C'_{J,k}(w_iw_f-w_{if})^{J-2k}[(w_i^2-1)(w_f^2-1)-(w_iw_f-w_{if})^2]^k, \quad \text{(A6)}$$

where $C_{j,k}$ and $C'_{J,k}$ are the numerical coefficients given by

$$C_{J,k}=(-1)^k\frac{(J!)^2}{(2J)!}\frac{(2J-2k)!}{k!(J-k)!(J-2k)!}, \quad \text{(A7)}$$

$$C'_{J,k}=(-1)^k2^{J-2k}\frac{(J!)^2}{(2J)!}\frac{J!}{(k!)^2(J-2k)!}. \quad \text{(A8)}$$

The expression (A5) is useful when considering the limit $v_i\to v$ in which $w_i\to 1$ and $w_{if}\to w_f$ (or as well $v_f\to v$ in which $w_f\to 1$ and $w_{if}\to w_i$). The expression (A6) is useful when considering the limit $v_f\to v_i$ in which $w_{if}\to 1$ and $w_f\to w_i$. Indeed, the $k$th term in Eq. (A5) or Eq. (A6) vanishes at order $k$, and only the $k=0$ term survives in the considered limit.

In this paper the matrix elements (A3) are all we need. However, as we shall see, the full expression of $\Pi^{(v)}$ can be deduced from these particular matrix elements. For brevity, we write this full expression for a particle at rest, in which case only the purely spatial components are nonvanishing.

The tensor $\Pi^{(v)}_{\mu_1,\ldots,\mu_n;\nu_1,\ldots,\nu_n}$ for an arbitrary four-velocity $v$ is readily obtained from the formula below by the substitutions

$$\delta_{i_r i_{r'}} \to -g_{\mu_r \mu_{r'}} + v_{\mu_r} v_{\mu_{r'}},$$
$$\delta_{i_s j_{s'}} \to -g_{\mu_s \nu_{s'}} + v_{\mu_s} v_{\nu_{s'}}, \tag{A9}$$
$$\delta_{j_t j_{t'}} \to -g_{\nu_t \nu_{t'}} + v_{\nu_t} v_{\nu_{t'}}.$$

The formula is

$$\Pi_{i_1\cdots i_n;j_1\cdots j_n} = \sum_{0\leqslant k\leqslant n/2} f_{n,k} \sum_{\substack{I,J\subset\{1\cdots n\}\\ |I|=|J|=2k}} \times\left(\sum_{\mathcal{J}\in\mathcal{P}_2(I)} \prod_{\{r,r'\}\in\mathcal{J}} \delta_{i_r i_{r'}}\right) \times\left(\sum_{\sigma\in\mathcal{B}(\mathbf{C}I,\mathbf{C}J)} \prod_{s\notin I} \delta_{i_s j_{\sigma(s)}}\right) \times\left(\sum_{\mathcal{J}'\in\mathcal{P}_2(J)} \prod_{\{t,t'\}\in\mathcal{J}'} \delta_{j_t j_{t'}}\right) \tag{A10}$$

with

$$f_{n,k} = (-1)^k 2^{2k} \frac{k!(2n-2k)!}{(n-k)!(2n)!}. \tag{A11}$$

In this formula (A10), $\mathbf{C}I$ and $\mathbf{C}J$ are the complementary sets in $\{1,2,\ldots,n\}$ of the subsets $I$ and $J$. For a set $X$, $\mathcal{P}_2(X)$ is the set of partitions of $X$ by two-element subsets (precisely, unordered partitions, or partitions as sets of subsets). For two sets $X$ and $Y$, $\mathcal{B}(X,Y)$ is the set of bijections $X\to Y$.

[There is a logical subtlety in the (important) terms $I=J=\emptyset$ in Eq. (A10); namely, one has

$$\sum_{\mathcal{J}\in\mathcal{P}_2(\emptyset)} \prod_{\{r,r'\}\in\mathcal{J}} \delta_{i_r,i_{r'}} = 1.$$

The reason is that the set (of sets) $\mathcal{P}_2(\emptyset)$ is not the empty set (else the sum would be 0), but is $\{\emptyset\}$. It contains the only element $\mathcal{J}=\emptyset$ (the empty set), and the product $\Pi_{\{r,r'\}\in\mathcal{J}}\delta_{i_r,i_{r'}}$ of an empty family is conventionally 1.]

In words, a term in Eq. (A10) is obtained as follows. Select an even number $2k$ of indices among the $i$'s and also among the $j$'s. Match the remaining $i$'s with the remaining $j$'s and include a factor $\delta_{ij}$ for each matched pair $(i, j)$. Divide the $2k$ selected $i$'s into pairs and include a factor $\delta_{ii'}$ for each pair $(i,i')$. Divide the $2k$ selected $j$'s into pairs and include a factor $\delta_{jj'}$ for each pair $(j,j')$. Different terms correspond to different such products of $\delta$'s. The lower rank ($n\leqslant 3$) tensors are written

$$\Pi_{i_1;j_1} = \delta_{i_1 j_1}, \tag{A12}$$

$$\Pi_{i_1,i_2;j_1,j_2} = \frac{1}{2}(\delta_{i_1j_1}\delta_{i_2j_2} + \delta_{i_1j_2}\delta_{i_2j_1}) - \frac{1}{3}\delta_{i_1i_2}\delta_{j_1j_2}, \tag{A13}$$

$$\begin{aligned}\Pi_{i_1,i_2,i_3;j_1,j_2,j_3} = {}& \frac{1}{6}(\delta_{i_1j_1}\delta_{i_2j_2}\delta_{i_3j_3} + \delta_{i_1j_1}\delta_{i_2j_3}\delta_{i_3j_2} + \delta_{i_1j_2}\delta_{i_2j_1}\delta_{i_3j_3} + \delta_{i_1j_2}\delta_{i_2j_3}\delta_{i_3j_1} + \delta_{i_1j_3}\delta_{i_2j_1}\delta_{i_3j_2} + \delta_{i_1j_3}\delta_{i_2j_2}\delta_{i_3j_1})\\ & - \frac{1}{15}(\delta_{i_1i_2}\delta_{i_3j_3}\delta_{j_1j_2} + \delta_{i_1i_2}\delta_{i_3j_2}\delta_{j_1j_3} + \delta_{i_1i_2}\delta_{i_3j_1}\delta_{j_2j_3} + \delta_{i_1i_3}\delta_{i_2j_3}\delta_{j_1j_2} + \delta_{i_1i_3}\delta_{i_2j_2}\delta_{j_1j_3} + \delta_{i_1i_3}\delta_{i_2j_1}\delta_{j_2j_3}\\ & + \delta_{i_2i_3}\delta_{i_1j_3}\delta_{j_1j_2} + \delta_{i_2i_3}\delta_{i_1j_2}\delta_{j_1j_3} + \delta_{i_2i_3}\delta_{i_1j_1}\delta_{j_2j_3}).\end{aligned} \tag{A14}$$

**Reduction to a three-dimensional problem**

We now turn to proofs of the above results. As a preliminary step, observe that the problem is reduced to a three-dimensional problem. Indeed, due to Lorentz covariance, it is enough to consider a particle at rest, namely, $v=(1,\vec{0})$. Then the transversity condition (3) above amounts to saying that $\varepsilon_{\mu_1\cdots\mu_J}=0$ if *one* of the indices is 0. Therefore, a polarization tensor is completely determined by its purely spatial components $\varepsilon_{i_1\cdots i_J}$, all the other components being zero. On these three-dimensional tensors, the conditions of symmetry (1) and of tracelessness (2) are written as:

(1′) Symmetry: $\varepsilon_{i_1\cdots i_J} = \varepsilon_{i_{\sigma(1)}\cdots i_{\sigma(J)}}$ for any permutation $\sigma$ of $1,\ldots,J$.

(2′) Tracelessness: $\Sigma_i \varepsilon_{iii_3\cdots i_J}=0$ (when $J\geqslant 2$).

The orthonormality conditions (A1) are written

$$\sum_{i_1\cdots i_J} \varepsilon^{(\lambda)}_{i_1\cdots i_J} (\varepsilon^{(\lambda')}_{i_1\cdots i_J})^* = \delta_{\lambda\lambda'}. \tag{A15}$$

The tensor $\Pi$ is also purely spatial and, according to Eq. (A2), is given by

$$\Pi_{i_1\cdots i_J;i'_1\cdots i'_J} = \sum_{\lambda=-J}^{J} \varepsilon^{(\lambda)}_{i_1\cdots i_J} (\varepsilon^{(\lambda)}_{i'_1\cdots i'_J})^*. \tag{A16}$$

But the preceding consideration identifies this spatial $\Pi$ as the projection operator, in the space of tensors of rank $J$, on the subspace of traceless symmetric tensors. Indeed, according to Eq. (A16), the tensor $\Sigma_{i'_1\cdots i'_J}\Pi_{i_1\cdots i_J;i'_1\cdots i'_J}T_{i'_1\cdots i'_J}$ is traceless symmetric for any tensor $T_{i_1\cdots i_J}$ and, according to Eqs. (A15) and (A16), one has

$$\sum_{i'_1\cdots i'_J}\Pi_{i_1\cdots i_J;i'_1\cdots i'_J}\varepsilon_{i'_1\cdots i'_J}=\varepsilon_{i_1\cdots i_J} \tag{A17}$$

for any traceless symmetric tensor $\varepsilon_{i_1\cdots i_J}$.

The problem of finding the projector on the polarization tensors is now reduced to the problem of finding the projector on the *spatial* symmetric traceless tensors.

### Deduction of the projector's particular matrix elements by standard methods of angular momentum coupling

Let us now turn to a proof of Eq. (A3). The space of rank $J$ tensors is just the tensor product of a number $J$ of the angular momentum 1 representation of the rotation group. The subspace of traceless symmetric tensors is just the subspace of angular momentum $J$, as can be understood since this subspace is used to describe the spin states of a particle of spin $J$.

Our problem is now reduced to the coupling of $J$ copies of the angular momentum 1 into a total angular momentum $J$. We now on use standard methods of angular momentum coupling.

The Clebsch-Gordan coefficients for the coupling of two angular momenta $J_1$ and $J_2$ to the maximum value $J_1+J_2$ has the following simple factorized form:

$$\langle J_1+J_2,M|J_1,J_2,M_1,M_2\rangle =\delta_{M,M_1+M_2}\frac{c(J_1,M_1)c(J_2,M_2)}{c(J_1+J_2,M)} \tag{A18}$$

with

$$c(J,M)=\frac{\sqrt{(2J)!}}{\sqrt{(J+M)!(J-M)!}}. \tag{A19}$$

The coupling coefficients of three angular momenta $J_1,J_2,J_3$ to the maximum value $J_1+J_2+J_3$, defined by

$$\langle J_1+J_2+J_3,M|J_1,J_2,J_3,M_1,M_2,M_3\rangle =\sum_{M'}\langle J_1+J_2+J_3,M|J_1+J_2,J_3,M',M_3\rangle \times\langle J_1+J_2,M'|J_1,J_2,M_1,M_2\rangle, \tag{A20}$$

are easily calculated from Eq. (A18):

$$\langle J_1+J_2+J_3,M|J_1,J_2,J_3,M_1,M_2,M_3\rangle =\delta_{M,M_1+M_2+M_3}\frac{c(J_1,M_1)c(J_2,M_2)c(J_3,M_3)}{c(J_1+J_2+J_3,M)}. \tag{A21}$$

Moreover, these coefficients do not depend on the particular order of coupling chosen in Eq. (A19) (first coupling $J_1$ and $J_2$, and then coupling the result to $J_3$).

By a simple recursive argument, one finds from Eq. (A19) that the coupling coefficients of $n$ angular momenta $J_1,\dots,J_n$ to the maximum value $J_1+\cdots+J_n$ are given by

$$\langle J_1+\cdots+J_n,M|J_1,\dots,J_n,M_1,\dots,M_n\rangle =\delta_{M,M_1+\cdots+M_n}\frac{c(J_1,M_1)\cdots c(J_n,M_n)}{c(J_1+\cdots+J_n,M_n)} \tag{A22}$$

and are independent of the order of the couplings. Recall that the $|J_1+\cdots+J_n,M\rangle$ are basis states of the $J_1+\cdots+J_n$ angular momentum subspace in the tensorial product of the $J_1\cdots J_n$ representation spaces of SU(2), and that the coefficient $\langle J_1+\cdots+J_n,M|J_1,\dots,J_n,M_1,\dots,M_n\rangle$ is the scalar product of the state $|J_1+\cdots+J_n,M\rangle$ with the basis state

$$|J_1,\dots,J_n,M_1,\dots,M_n\rangle=|J_1,M_1\rangle\otimes\cdots\otimes|J_n,M_n\rangle \tag{A23}$$

in the tensorial product space.

Now we may take the case of interest to us, $J_1=\cdots=J_n=1$, with the $J=1$ representation of SU(2) in the form of the ordinary rotations in $\mathbb{C}^3$ space (complexified ordinary three-dimensional space). The tensorial product space is just the space of tensors of order $n$, and the $J_1+\cdots+J_n=n$ subspace is just the subspace of traceless symmetric tensors. The states $|1,\dots,1,M_1,\dots,M_n\rangle$ *are the tensorial products of standard basis vectors* $|1,M\rangle$ of $\mathbb{C}^3$, and the states $|n,M\rangle$ constitute a standard basis of symmetric tensors. We are interested in the scalar product of the tensors $|n,M\rangle$ with the tensors $\vec{x}^{\otimes n}$ $[(\vec{x}^{\otimes n})_{i_1\cdots i_n}=x_{i_1}\cdots x_{i_n}]$ for any $\vec{x}\in\mathbb{C}^3$. Therefore, we have to expand the tensors $\vec{x}^{\otimes n}$ in the basis $|1,\dots,1,M_1,\dots,M_n\rangle$.

The qualifier "standard" above means in conformity to the standard definition of the Clebsch-Gordan coefficients. The standard basis of $\mathbb{C}^3$ is

$$|1,1\rangle=\vec{f}_1=-\frac{1}{\sqrt{2}}(\vec{e}_1+i\vec{e}_2),$$

$$|1,0\rangle=\vec{f}_0=\vec{e}_3,$$

$$|1,-1\rangle=\vec{f}_{-1}=\frac{1}{\sqrt{2}}(\vec{e}_1-i\vec{e}_2), \tag{A24}$$

where $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ is the Cartesian basis. Then a vector $\vec{x}=x_1\vec{e}_1+x_2\vec{e}_2+x_3\vec{e}_3\in\mathbb{C}^3$ can be written

$$\vec{x}=-\frac{x_1-ix_2}{\sqrt{2}}\vec{f}_1+\frac{x_1+ix_2}{\sqrt{2}}\vec{f}_{-1}+x_3\vec{f}_0 \tag{A25}$$

and the tensor $(\vec{x})^{\otimes n}$ is expanded as

$$(\vec{x})^{\otimes n} = \sum_{k,k'} \frac{n!}{k!k'!(n-k-k')!}(-1)^k \times \left(\frac{x_1 - ix_2}{\sqrt{2}}\right)^k \left(\frac{x_1 + ix_2}{\sqrt{2}}\right)^{k'} (x_3)^{n-k-k'} \times \mathrm{Sym}(\vec{f}_1)^{\otimes k} \otimes (\vec{f}_{-1})^{\otimes k'} \otimes (\vec{f}_0)^{\otimes n-k-k'} \tag{A26}$$

where Sym is the projector on symmetric tensors

$$(\mathrm{Sym}\ \ T)_{i_1 \cdots i_n} = \frac{1}{n!} \sum_{\sigma} T_{i_{\sigma(1)} \cdots i_{\sigma(n)}}.$$

Actually, equipped with the symmetrized product, the symmetric tensors constitute a commutative algebra, so that the formula (A26) is just obtained by multinomial expansion.

Then we have

$$\langle n,M|(\vec{x})^{\otimes n}\rangle = \sum_{k,k'} \frac{n!}{k!k'!(n-k-k')!}(-1)^k \times \left(\frac{x_1 - ix_2}{\sqrt{2}}\right)^k \left(\frac{x_1 + ix_2}{\sqrt{2}}\right)^{k'} \times (x_3)^{n-k-k'} \langle n,M|(\vec{f}_1)^{\otimes k} \otimes (\vec{f}_{-1})^{\otimes k'} \otimes (\vec{f}_0)^{\otimes n-k-k'}\rangle \tag{A27}$$

where the Sym operator has been dropped because the coupling coefficients do not depend on the order of the couplings. Formula (A22) now readily gives

$$\langle n,M|(\vec{x})^{\otimes n}\rangle = \sum_{k,k'} \frac{n!}{k!k'!(n-k-k')!}(-1)^k \times \left(\frac{x_1 - ix_2}{\sqrt{2}}\right)^k \left(\frac{x_1 + ix_2}{\sqrt{2}}\right)^{k'} (x_3)^{n-k-k'} \times \delta_{M,k-k'} \frac{c(1,1)^k c(1,0)^{n-k-k'} c(1,-1)^{k'}}{c(n,M)}. \tag{A28}$$

An easy calculation (just undoing the multinomial expansion) gives the following generating function for these $\langle n,M|(\vec{x})^{\otimes n}\rangle$:

$$\sum_{M=-n}^{n} c(n,M)\langle n,M|(\vec{x})^{\otimes n}\rangle s^M = \left[ -c(1,1)\frac{x_1 - ix_2}{\sqrt{2}} s + c(1,0)x_3 - c(1,-1)\frac{x_1 + ix_2}{\sqrt{2}} s^{-1}\right]^n. \tag{A29}$$

According to formula (A19), we have

$$c(n,M) = \frac{\sqrt{(2n)!}}{\sqrt{(n+M)!(n-M)!}},$$

$$c(1,1)=1, \quad c(1,0)=\sqrt{2}, \quad c(1,-1)=1. \tag{A30}$$

Therefore

$$\sum_{M=-n}^{n} \frac{\sqrt{(2n)!}}{\sqrt{(n+M)!(n-M)!}} \langle n,M|(\vec{x})^{\otimes n}\rangle s^M = 2^{n/2}\left( -\frac{x_1 - ix_2}{2}s + x_3 + \frac{x_1 + ix_2}{2}s^{-1}\right)^n. \tag{A31}$$

Comparing this to the generating function for the solid spherical harmonics $\mathcal{Y}_L^M(\vec{x}) = |\vec{x}|^L Y_L^M(\hat{x})$, which is

$$\sum_{M=-L}^{L} \frac{(L)!}{\sqrt{(L-M)!(L+M)!}} \mathcal{Y}_L^M(\vec{r}) s^M = \frac{\sqrt{2L+1}}{\sqrt{4\pi}} \left( \frac{x_1 - ix_2}{2} s^{-1} + x_3 - \frac{x_1 + ix_2}{2} s\right)^L, \tag{A32}$$

we arrive at the fundamental result

$$\langle n,M|(\vec{x})^{\otimes n}\rangle = 2^{n/2} \frac{n!}{\sqrt{(2n+1)!}} \sqrt{4\pi} \mathcal{Y}_n^M(\vec{x})^*. \tag{A33}$$

From this we compute the matrix element $\langle (\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle$ of the sought projector $\Pi_n$ (on the traceless symmetric tensors). One has

$$\langle (\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle = \sum_{M=-n}^{n} \langle n,M|(\vec{y})^{\otimes n}\rangle^* \langle n,M|(\vec{x})^{\otimes n}\rangle = 2^n \frac{(n!)^2}{(2n+1)!} 4\pi \sum_{M=-n}^{n} \mathcal{Y}_n^M(\vec{y}) \mathcal{Y}_n^M(\vec{x})^*, \tag{A34}$$

and using

$$\sum_{M=-n}^{n} \mathcal{Y}_n^M(\vec{x}) \mathcal{Y}_n^M(\vec{y})^* = \frac{2n+1}{4\pi} |\vec{x}|^n |\vec{y}|^n P_n\left(\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}\right) \tag{A35}$$

one readily obtains

$$\langle (\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle = 2^n \frac{(n!)^2}{(2n)!} |\vec{x}|^n |\vec{y}|^n P_n\left(\frac{\vec{x}\cdot\vec{y}}{|\vec{x}||\vec{y}|}\right). \tag{A36}$$

One may then introduce explicit expressions for the Legendre polynomials $P_n$:

$$P_n(x) = \frac{1}{2^n} \sum_{0 \leqslant k \leqslant n/2} (-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} x^{n-2k}, \tag{A37}$$

$$P_n(x)=\sum_{0\leqslant k\leqslant n/2}(-1)^k\frac{1}{2^{2k}}\frac{n!}{(k!)^2(n-2k)!}\times x^{n-2k}(1-x^2)^k, \tag{A38}$$

and we obtain the following explicit expressions for $\langle(\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle$:

$$\langle(\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle=\sum_{0\leqslant k\leqslant n/2}C_{n,k}(\vec{x}^2)^k(\vec{x}\cdot\vec{y})^{n-2k}(\vec{y}^2)^k$$
$$=\sum_{0\leqslant k\leqslant n/2}C'_{n,k}(\vec{x}\cdot\vec{y})^{n-2k}\times[\vec{x}^2\vec{y}^2-(\vec{x}\cdot\vec{y})^2]^k, \tag{A39}$$

$$C_{n,k}=(-1)^k\frac{(n!)^2}{(2n)!}\frac{(2n-2k)!}{k!(n-k)!(n-2k)!},$$

$$C'_{n,k}=(-1)^k 2^{n-2k}\frac{(n!)^2}{(2n)!}\frac{n!}{(k!)^2(n-2k)!}. \tag{A40}$$

To go back to an arbitrary velocity $v$ and obtain Eq. (A3), just set $\vec{x}=\vec{v}_i$, $\vec{y}=\vec{v}_f$, and use the following formulas:

$$\vec{v}_i^2=(v_i\cdot v)^2-v_i^2=w_i^2-1,$$
$$\vec{v}_f^2=(v_f\cdot v)^2-v_f^2=w_f^2-1,$$
$$\vec{v}_i\cdot\vec{v}_f=(v_i\cdot v)(v_f\cdot v)-v_i\cdot v_f=w_iw_f-w_{if}. \tag{A41}$$

**Deduction of the projector itself from its particular matrix elements**

We now present a deduction of $\Pi_{i_1,\dots,i_n;j_1,\dots,j_n}$ from the matrix elements $\langle(\vec{y})^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\rangle$. To see how to proceed, let us consider a multilinear function $F(\vec{x}_1,\dots,\vec{x}_n)$, which is *symmetric* in the permutations of its $n$ vector variables $\vec{x}_1,\dots,\vec{x}_n$. Then it can be recovered from its diagonal values $F((\vec{x})_n)=F(\vec{x},\dots,\vec{x})$ by the following formula:

$$F(\vec{x}_1,\dots,\vec{x}_n)=\frac{(-1)^n}{n!}\sum_{\substack{s_1,\dots,s_n\\0\leqslant s_i\leqslant 1}}(-1)^{s_1+\cdots+s_n}\times F((s_1\vec{x}_1+\cdots+s_n\vec{x}_n)n). \tag{A42}$$

Indeed, expanding $F((s_1\vec{x}_1+\cdots+s_n\vec{x}_n)_n)$ by multinearity and collecting terms equal by symmetry, one has

$$F((s_1\vec{x}_1+\cdots+s_n\vec{x}_n)_n)=\sum_{\substack{q_1,\dots,q_n\\q_1+\cdots+q_n=n}}\frac{n!}{q_1!\cdots q_n!}s_1^{q_1}\cdots s_n^{q_n}F((\vec{x}_1)_{q_1},\dots,(\vec{x}_n)q_n), \tag{A43}$$

where the notation $(\vec{x}_i)_{q_i}$ [also used in Eq. (A42)] stands for the $q_i$-uple $(\vec{x}_i,\dots,\vec{x}_i)$. A term in Eq. (A43) with *some* $q_i$ vanishing gives no contribution to Eq. (A42) because it does not depend on $s_i$, and the corresponding $s_i=0$ and $s_i=1$ terms in Eq. (A42) cancel. Then, since $q_1+\cdots+q_n=n$, the only term of Eq. (A43) contributing to Eq. (A42) is $q_1=\cdots=q_n=1$. The right-hand side of Eq. (A42) is therefore equal to

$$\frac{(-1)^n}{n!}\sum_{\substack{s_1,\dots,s_n\\0\leqslant s_i\leqslant 1}}(-1)^{s_1+\cdots+s_n}n!s_1\cdots s_nF(\vec{x}_1,\dots,\vec{x}_n)=F(\vec{x}_1,\dots,\vec{x}_n).$$

Using Eq. (A42), we can now deduce $\Pi_{i_1,\dots,i_n;j_1,\dots,j_n}$ from the matrix elements in two steps. As a first step, let us apply formula (A42) to the multilinear symmetric function

$$(\vec{y}_1,\dots,\vec{y}_n)\to\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle \tag{A44}$$

with $\vec{x}$ fixed. This gives

$$\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x}^{\otimes n}\rangle=\frac{(-1)^n}{n!}\sum_{\substack{s_1,\dots,s_n\\0\leqslant s_i\leqslant 1}}(-1)^{s_1+\cdots+s_n}\times\left\langle\left(\sum_i s_i\vec{y}_i\right)^{\otimes n}|\Pi_n|(\vec{x})^{\otimes n}\right\rangle, \tag{A45}$$

or, using Eq. (A39),

$$\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle=\frac{(-1)^n}{n!}\sum_{0\leqslant k\leqslant n/2}C_{n,k}(\vec{x}^2)^k\times\sum_{\substack{s_1,\dots,s_n\\0\leqslant s_i\leqslant 1}}(-1)^{s_1+\cdots+s_n}\left(\sum_i s_i(\vec{y}_i\cdot\vec{x})\right)^{n-2k}\times\left[\left(\sum_i s_i\vec{y}_i\right)^2\right]^k. \tag{A46}$$

Then we work out the multinomial expansions

$$\left(\sum_i s_i(\vec{y}_i\cdot\vec{x})\right)^{n-2k}=\sum_{\substack{u_1,\dots,u_n\geqslant 0\\u_1+\cdots+u_n=n-2k}}\frac{(n-2k)!}{u_1!\cdots u_n!}\times\prod_{i=1}^n(s_i)^{u_i}(\vec{y}_i\cdot\vec{x})^{u_i}, \tag{A47}$$

$$\left[\left(\sum_i s_i\vec{y}_i\right)^2\right]^k=\left(\sum_{1\leqslant i,i'\leqslant n}s_is_{i'}(\vec{y}_i\cdot\vec{y}_{i'})\right)^k=\sum_{\substack{v_{11},v_{12},\dots,v_{n-1,n},v_{nn}\geqslant 0\\v_{11}+v_{12}+\cdots+v_{nn}=k}}\frac{k!}{v_{11}!v_{12}!\cdots v_{nn}!}\times\prod_{i,i'=1}^n(s_is_{i'})^{v_{ii'}}(\vec{y}_i\cdot\vec{y}_{i'})^{v_{ii'}}. \tag{A48}$$

Using this in Eq. (A46) and collecting the powers of $s_i$, we have

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle = \frac{(-1)^n}{n!}\sum_{0\leq k\leq n/2} C_{n,k}(\vec{x}^2)^k \times \sum_{\substack{u_1,\ldots,u_n\geq 0\\ u_1+\cdots+u_n=n-2k}} \sum_{\substack{v_{11},v_{12},\ldots,v_{n-1,n},v_{nn}\geq 0\\ v_{11}+v_{12}+\cdots+v_{nn}=k}} \frac{(n-2k)!}{u_1!\cdots u_n!} \times \frac{k!}{v_{11}!v_{12}!\cdots v_{nn}!}\sum_{\substack{s_1,\ldots,s_n\\ 0\leq s_i\leq 1}}(-1)^{s_1+\cdots+s_n}\left[\prod_{i=1}^n (s_i)^{p_i}\right] \times\left[\prod_{i=1}^n(\vec{y}_i\cdot\vec{x})^{u_i}\right]\left[\prod_{i,i'=1}^n(\vec{y}_i\cdot\vec{y}_{i'})^{v_{ii'}}\right], \tag{A49}$$

where the exponent $p_i$ of $s_i$ is

$$p_i=u_i+\sum_{i'=1}^n (v_{i'i}+v_{ii'}). \tag{A50}$$

Notice now that, for values of the $u_i$'s and of the $v_{ii'}$'s such that some exponent $p_{i_0}$ vanishes, the $s_{i_0}=0$ and the $s_{i_0}=1$ terms cancel. Since, according to the constraints on the $u_i$'s and the $v_{ii'}$'s, one has

$$\sum_{i=1}^n p_i=\sum_{i=1}^n u_i+2\sum_{i,i'=1}^n v_{ii'}=n, \tag{A51}$$

we are left with the values of the $u_i$'s and of the $v_{ii'}$'s such that $p_1=\cdots=p_n=1$. These values can then be only 0 or 1, so that $u_i!=1$ and $v_{ii'}!=1$. Moreover, we then have a factor $s_1\cdots s_n$, so that only $s_1=\cdots=s_n=1$ contributes. So Eq. (A49) reduces to

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle = \sum_{0\leq k\leq n/2}\frac{k!(n-2k)!}{n!}C_{n,k}(\vec{x}^2)^k \times \sum_{\substack{u_1,\ldots,u_n,v_{11},v_{11},v_{12},\ldots,v_{n-1,n},v_{nn}\geq 0\\ u_1+\cdots+u_n=n-2k\\ u_i+v_{1i}+\cdots+v_{ni}+v_{i1}+\cdots+v_{in}=1}}\left[\prod_{i=1}^n(\vec{y}_i\cdot\vec{x})^{u_i}\right] \times\left[\prod_{i,i'=1}^n(\vec{y}_i\cdot\vec{y}_{i'})^{v_{ii'}}\right] \tag{A52}$$

where we have dropped the constraint $\Sigma_{i,i'=1}^n v_{ii'}=k$ since it is implied by the remaining constraints.

Since the $u_i$'s take only the values 0 or 1, we can replace $(u_1,\ldots,u_n)$, as the summation variable, by subsets $I$ of $\{1,...,n\}$. It will be convenient to use the subset related to $(u_1,\ldots,u_n)$ by $I=\{i|u_i=0\}$. The constraint $u_1+\cdots+u_n=n-2k$ is translated into the constraint $|I|=2k$, and formula (A52) becomes

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle = \sum_{0\leq k\leq n/2}\frac{k!(n-2k)!}{n!}C_{n,k}(\vec{x}^2)^k\sum_{\substack{I\subset\{1\cdots n\}\\ |I|=2k}}\left[\prod_{i\notin I}(\vec{y}_i\cdot\vec{x})\right] \times\sum_{\substack{v_{11},v_{11},v_{12},\ldots,v_{n-1,n},v_{nn}\geq 0\\ v_{1i}+\cdots v_{ni}+v_{i1}+\cdots+v_{in}=0\,(i\notin I)\\ v_{1i}+\cdots+v_{ni}+v_{i1}+\cdots+v_{in}=1\,(i\in I)}}\left[\prod_{i,i'\in I}(\vec{y}_i\cdot\vec{y}_{i'})^{v_{ii'}}\right]. \tag{A53}$$

Consider now the constraints on the $v_{ii'}$'s. If $i\notin I$ or $i'\notin I$, we have $v_{ii'}=0$. We are left with the $v_{ii'}$ for $i,i'\in I$, constrained by

$$\sum_{i'\in I}(v_{ii'}+v_{i'i})=1 \quad \text{for any } i\in I. \tag{A54}$$

Notice that $v_{ii}=0$, since $v_{ii}$ occurs twice in this sum. Let us then replace the $v_{ii'}$'s, as summation variable, by the set $\mathcal{J}$ of two-element subsets of $I$ related to the $v_{ii'}$'s by

$$\mathcal{J}=\{\{i,i'\}|v_{ii'}+v_{i'i}=1). \tag{A55}$$

For any $i\in I$, there is, according to Eq. (A54), one and only one $i'\in I$ such that $\{i,i'\}\in\mathcal{J}$. In other words, $\mathcal{J}$ belongs to the set $\mathcal{P}_2(I)$ of partitions of $J$ by two-element subsets. Conversely, if $\mathcal{J}\in\mathcal{P}_2(I)$, the values of $v_{ii'}+v_{i'i}$ defined by Eq. (A55), namely,

$$v_{ii'}+v_{i'i}=1 \quad \text{if } \{i,i'\}\in\mathcal{J}, \quad v_{ii'}+v_{i'i}=0 \quad \text{if } \{i,i'\}\notin\mathcal{J}, \tag{A56}$$

do satisfy the constraints (A54), since for any $i\in I$, there is one and only one $i'\in I$ such that $\{i,i'\}\in\mathcal{J}$. Now, Eq. (A56) does not determine completely the values of the $v_{ii'}$'s. When $\{i,i'\}\notin\mathcal{J}$ we must have $v_{ii'}=v_{i'i}=0$, but when $\{i,i'\}\in\mathcal{J}$ we have two solutions: $v_{ii'}=1$, $v_{i'i}=0$ and $v_{ii'}=0$, $v_{i'i}=1$. Since $|\mathcal{J}|=k$, we have in all $2^k$ values of the $v_{ii'}$'s corresponding to each $\mathcal{J}\in\mathcal{P}_2(I)$. However, values of the $v_{ii'}$'s corresponding to the same $\mathcal{J}$ give equal terms in Eq. (A53) and, lumping these terms together, the formula (A53) becomes

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|(\vec{x})^{\otimes n}\rangle = \sum_{0\leq k\leq n/2} 2^k\frac{k!(n-2k)!}{n!}C_{n,k} \times(\vec{x}^2)^k\sum_{\substack{I\subset\{1\cdots n\}\\|I|=2k}}\left[\prod_{i\notin I}(\vec{y}_i\cdot\vec{x})\right] \times\left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)}\prod_{\{i,i'\}\in\mathcal{J}}(\vec{y}_i\cdot\vec{y}_{i'})\right]. \tag{A57}$$

As a second (and last) step, let us apply formula (A42) to the multilinear symmetric function

$$(\vec{x}_1,\ldots,\vec{x}_n)\to\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle \tag{A58}$$

with $(\vec{y}_1,\ldots,\vec{y}_n)$ fixed. This gives

$$\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle = \frac{(-1)^n}{n!}\sum_{\substack{s_1,\ldots,s_n\\0\leq s_i\leq 1}}(-1)^{s_1+\cdots+s_n} \times\left\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\left(\sum_j s_j\vec{x}_j\right)^{\otimes n}\right\rangle \tag{A59}$$

or, using Eq. (A57),

$$\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle = \frac{(-1)^n}{n!}\sum_{0\leq k\leq n/2}2^k\frac{k!(n-2k)!}{n!}C_{n,k}\sum_{\substack{I\subset\{1\cdots n\}\\|I|=2k}}\left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)}\prod_{\{i,i'\}\in\mathcal{J}}(\vec{y}_i\cdot\vec{y}_{i'})\right] \times\sum_{\substack{s_1,\ldots,s_n\\0\leq s_j\leq 1}}(-1)^{s_1+\cdots+s_n}\left[\left(\sum_j s_j\vec{x}_j\right)^2\right]^k\left[\prod_{i\notin I}\left(\sum_j s_j(\vec{y}_i\cdot\vec{x}_j)\right)\right]. \tag{A60}$$

Using the expansions

$$\prod_{i\notin I}\left(\sum_j s_j(\vec{y}_i\cdot\vec{x}_j)\right) = \sum_{\substack{u_{ij}\geq 0(i\notin I,1\leq j\leq n)\\u_{i1}+\cdots+u_{in}=1}}\left[\prod_{\substack{i\notin I\\1\leq j\leq n}}(s_j)^{u_{ij}}(\vec{y}_i\cdot\vec{x}_j)^{u_{ij}}\right] \tag{A61}$$

$$\left[\left(\sum_j s_j\vec{x}_j\right)^2\right]^k = \left(\sum_{1\leq j,j'\leq n}s_js_{j'}(\vec{x}_j\cdot\vec{x}_{j'})\right)^k = \sum_{\substack{v_{11},v_{12},\ldots,v_{n-1,n},v_{nn}\geq 0\\v_{11}+v_{12}+\cdots+v_{nn}=k}}\frac{k!}{v_{11}!v_{12}!\cdots v_{nn}!}\left[\prod_{j,j'=1}^n(s_js_{j'})^{v_{jj'}}(\vec{x}_j\cdot\vec{x}_{j'})^{v_{jj'}}\right] \tag{A62}$$

in Eq. (A60) and collecting the powers of the $s_i$, we have

$$\langle\vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle = \frac{(-1)^n}{n!}\sum_{0\leq k\leq n/2}2^k\frac{k!(n-2k)!}{n!}C_{n,k}\sum_{\substack{I\subset\{1\cdots n\}\\|I|=2k}}\left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)}\prod_{\{i,i'\}\in\mathcal{J}}(\vec{y}_i\cdot\vec{y}_{i'})\right] \times\sum_{\substack{u_{ij}\geq 0(i\notin I,1\leq j\leq n)\\u_{i1}+\cdots+u_{in}=1}}\sum_{\substack{v_{11},v_{12},\ldots,v_{n-1,n},v_{nn}\geq 0\\v_{11}+v_{12}+\cdots+v_{nn}=k}}\frac{k!}{v_{11}!v_{12}!\cdots v_{nn}!}\sum_{\substack{s_1,\ldots,s_n\\0\leq s_j\leq 1}}(-1)^{s_1+\cdots+s_n}\left[\prod_{j=1}^n(s_j)^{p_j}\right] \times\left[\prod_{\substack{i\notin I\\1\leq j\leq n}}(\vec{y}_i\cdot\vec{x}_j)^{u_{ij}}\right]\left[\prod_{j,j'=1}^n(\vec{x}_j\cdot\vec{x}_{j'})^{v_{jj'}}\right], \tag{A63}$$

where the exponent $p_j$ of $s_j$ is

$$p_j=\sum_{i\notin I}u_{ij}+\sum_{j'=1}^n(v_{j'j}+v_{jj'}). \tag{A64}$$

According to the constraints on the $u_{ij}$'s and the $v_{jj'}$'s, one has

$$\sum_{j=1}^{n} p_j = \sum_{\substack{i\notin I\\ 1\leq j\leq n}} u_{ij} + 2\sum_{j,j'=1}^{n} v_{jj'} = n, \tag{A65}$$

and, by the same arguments following Eq. (A49), one sees that only the terms $p_1=\cdots=p_n=1$, $s_1=\cdots=s_n=1$ contributes to Eq. (A63). So Eq. (A63) reduces to

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle = \sum_{0\leq k\leq n/2} 2^k \frac{(k!)^2(n-2k)!}{(n!)^2} C_{n,k} \sum_{\substack{I\subset\{1\cdots n\}\\ |I|=2k}} \left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)} \prod_{\{i,i'\}\in\mathcal{J}} (\vec{y}_i\cdot\vec{y}_{i'})\right]$$
$$\times \sum_{\substack{u_{ij},v_{jj'}\geq 0(i\notin I,1\leq j,j'\leq n)\\ \Sigma_{1\leq j\leq n}u_{ij}=1(i\notin I)\\ \Sigma_{i\notin I}u_{ij}+\Sigma_{1\leq j'\leq n}(v_{jj'}+v_{j'j})=1(1\leq j\leq n)}} \left[\prod_{\substack{i\notin I\\ 1\leq j\leq n}} (\vec{y}_i\cdot\vec{x}_j)^{u_{ij}}\right]\left[\prod_{j,j'=1}^{n} (\vec{x}_j\cdot\vec{x}_{j'})^{v_{jj'}}\right]. \tag{A66}$$

Consider the constraints on the $u_{ij}$'s:

$$\sum_{j=1}^{n} u_{ij}=1 \quad \text{for all } i\notin I, \quad \sum_{i\notin I} u_{ij}\leq 1 \quad \text{for all } 1\leq j\leq n. \tag{A67}$$

The solutions of Eq. (A67) are in one-to-one correspondence with the set of injective maps $v$: $\mathbf{C}I\rightarrow\{1,...,n\}$ by

$$u_{ij}=1 \quad \text{if } j=v(i), \quad u_{ij}=0 \quad \text{if } j\neq v(i). \tag{A68}$$

Indeed, the first constraint (A67) says that to each $i\notin I$ corresponds one and only one $j=v(i)$ such that $u_{ij}=1$, and this defines a map $v$: $\mathbf{C}I\rightarrow\{1,...,n\}$. The second one says that, for each $j$, there is at most one $i\notin I$ such that $u_{ij}=1$, and this means that the map $v$ is injective.

We can therefore replace the $u_{ij}$'s, as summation variables, by the injective maps $v$: $\mathbf{C}I\rightarrow\{1,...,n\}$. Furthermore, it will be convenient to replace $v$ by the pair $(J,\sigma)$ where $J=\mathbf{C}v(\mathbf{C}I)$ is the complementary subset of the image of $v$, and $\sigma$: $\mathbf{C}I\rightarrow\mathbf{C}J$ is the bijective map induced by $v$. Then formula (A68) becomes

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle$$
$$= \sum_{0\leq k\leq n/2} 2^k \frac{(k!)^2(n-2k)!}{(n!)^2} C_{n,k}$$
$$\times \sum_{\substack{I,J\subset\{1\cdots n\}\\ |I|=|J|=2k}} \left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)} \prod_{\{i,i'\}\in\mathcal{J}} (\vec{y}_i\cdot\vec{y}_{i'})\right]$$
$$\times\left[\sum_{\sigma\in\mathcal{B}(CI,CJ)} \prod_{i\notin I} (\vec{y}_i\cdot\vec{x}_{\sigma(i)})\right]$$
$$\times \sum_{\substack{v_{jj'}\geq 0(1\leq j,j'\leq n)\\ \Sigma_{1\leq j'\leq n}(v_{jj'}+v_{j'j})=0(j\notin J)\\ \Sigma_{1\leq j'\leq n}(v_{jj'}+v_{j'j})=1(j\in J)}} \left[\prod_{j,j'=1}^{n} (\vec{x}_j\cdot\vec{x}_{j'})v_{jj'}\right]. \tag{A69}$$

This last summation on the $v_{jj'}$'s is exactly the same as the summation on the $v_{ii'}$'s in formula (A53), and is treated in the same way. It is replaced by a summation on the partitions $\mathcal{J}'\in\mathcal{P}_2(J)$ of $J$ by two-element subsets. To each $\mathcal{J}'$ correspond $2^k$ solutions of the $v_{jj'}$ constraints, which give equal terms in Eq. (A69). Finally, we obtain

$$\langle \vec{y}_1\otimes\cdots\otimes\vec{y}_n|\Pi_n|\vec{x}_1\otimes\cdots\otimes\vec{x}_n\rangle$$
$$= \sum_{0\leq k\leq n/2} 2^k \frac{(k!)^2(n-2k)!}{(n!)^2} C_{n,k} \sum_{\substack{I,J\subset\{1\cdots n\}\\ |I|=|J|=2k}}$$
$$\times\left[\sum_{\mathcal{J}\in\mathcal{P}_2(I)} \prod_{\{i,i'\}\in\mathcal{J}} (\vec{y}_i\cdot\vec{y}_{i'})\right]$$
$$\times\left[\sum_{\sigma\in\mathcal{B}(CI,CJ)} \prod_{i\notin I} (\vec{y}_i\cdot\vec{x}_{\sigma(i)})\right]$$
$$\times\left[\sum_{\mathcal{J}'\in\mathcal{P}_2(I)} \prod_{\{j,j'\}\in\mathcal{J}'} (\vec{x}_j\cdot\vec{x}_{j'})\right]. \tag{A70}$$

Formula (A10) is just formula (A70) with

$$(\vec{y}_1,\ldots,\vec{y}_n)=(\vec{e}_{i_1},\ldots,\vec{e}_{i_n}), \quad (\vec{x}_1,\ldots,\vec{x}_n)=(\vec{e}_{j_1},\ldots,\vec{e}_{j_n})$$

where $(\vec{e}_1,\vec{e}_2,\vec{e}_3)$ is the Cartesian basis of $\mathbb{R}^3$.

**Direct proof of the expression of the projector**

The expression (A10) for the projector $\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ on the symmetric traceless tensors has been obtained by rather lengthy and indirect arguments. However, once expression (A10) is known, it becomes possible to verify directly that it gives the sought projector. We do this now, obtaining a new proof of Eq. (A10). This new proof goes on, without added complications, for an arbitrary dimension $D$ of space.

We thus consider now Eq. (A10) as a *tentative formula*, with unknown coefficients $f_{n,k}$. Let us enumerate the conditions for the $\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ to be the components of the projector $\Pi_n$ on the subspace of the symmetric traceless tensors (in the space of all $n$-rank tensors).

First, the image by $\Pi_n$ of any tensor must be in the projection subspace. For the components, this means that for fixed $(j_1,\ldots,j_n)$, the $n$-rank tensor $\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ is symmetric traceless. Therefore we have the two conditions.

(1) $\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$ is symmetric with respect to the $i$'s;

$$(2)\quad \sum_{i_1,i_2=1}^{D} \delta_{i_1 i_2}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}=0$$

$$\text{for all } i_3,\ldots,i_n,j_1,\ldots,j_n. \tag{A71}$$

Next, $\Pi_n$ must transform into itself any tensor in the projection subspace. This gives the following third condition:

$$(3)\quad \sum_{j_1,\ldots,j_n=1}^{D} \Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}\varepsilon_{j_1,\ldots,j_n}=\varepsilon_{i_1,\ldots,i_n} \tag{A72}$$

for any symmetric traceless tensor $\varepsilon_{i_1,\ldots,i_n}$.

Conditions 1, 2, and 3 say that $\Pi_n$ is *some* projector on the symmetric traceless tensors. To specify it completely, we must add that $\Pi_n$ is the *orthogonal* projector (i.e., it annihilates any tensor orthogonal to all symmetric traceless tensors). It is equivalent to saying that $\Pi_n$ is a symmetric (or Hermitian) operator. This gives a fourth and last condition:

$$(4)\quad \Pi_{i_1,\ldots,i_n;j_1,\ldots j_n}=\Pi_{j_1,\ldots,j_n;i_1,\ldots,i_n}. \tag{A73}$$

Let us see now if we can satisfy these conditions with the formula (A10).

The conditions 1 and 4 are easy. They are satisfied independently of the coefficients $f_{n,k}$. The symmetry with respect to the $i$'s stems from the fact that only the *set* $\{1,\ldots,n\}$ of the numbers which specify the $i$'s enters in Eq. (A10). The symmetry with respect to the exchange of the $i$'s and $j$'s is also clearly satisfied by Eq. (A10).

Condition 3 is not difficult to deal with. Notice that any term in Eq. (A10) containing a factor $\delta_{j_t j_{t'}}$ gives no contribution to the left-hand side of Eq. (A72). This is due to the fact that $\varepsilon_{i_1,\ldots,i_n}$ is symmetric traceless, so that

$$\sum_{j_t,j_{t'}=1}^{D} \delta_{j_t j_{t'}}\varepsilon_{j_1,\ldots,j_n}=0. \tag{A74}$$

The only terms in Eq. (A10) without any factor $\delta_{j_t j_{t'}}$ are the ones with $J$ empty. But $J=\emptyset$ implies $k=0$ and $I=\emptyset$. So, with the trial formula (A10), condition 3 takes the form

$$f_{n,0}\sum_{j_1,\ldots,j_n=1}^{D} \sum_{\sigma\in\mathcal{P}_n} \prod_{s=1}^{n} \delta_{i_s j_{\sigma(s)}}\varepsilon_{j_1,\ldots,j_n}=\varepsilon_{i_1,\ldots,i_n}, \tag{A75}$$

or, summing the $j_i$'s,

$$f_{n,0}\sum_{\sigma\in\mathcal{P}_n} \varepsilon_{i_{\sigma(1)},\ldots,i_{\sigma(n)}}=\varepsilon_{i_1,\ldots,i_n}, \tag{A76}$$

where $\mathcal{P}_n$ is the set of permutations of the set $\{1,\ldots,n\}$. Taking into account the symmetry of $\varepsilon_{i_1,\ldots,i_n}$, and since $|\mathcal{P}_n|=n!$, this is the same as

$$n!\,f_{n,0}\varepsilon_{i_1,\ldots,i_n}=\varepsilon_{i_1,\ldots,i_n}. \tag{A77}$$

Therefore, condition 3 just fixes $f_{n,k}$ for $k=0$:

$$f_{n,0}=\frac{1}{n!}. \tag{A78}$$

Condition 2 is the hard one. The first thing to be done is to rewrite Eq. (A10) in a form where the occurrences of the indices $i_1$ and $i_2$ are explicit. The result of this step is formula (A83) below. To alleviate the formulas, let us introduce the following notation:

$$X(I)=\sum_{\mathcal{J}\in\mathcal{P}_2(I)} \prod_{\{r,r'\}\in\mathcal{J}} \delta_{i_r,i_{r'}},$$

$$Y(J)=\sum_{\mathcal{J}'\in\mathcal{P}_2(J)} \prod_{\{t,t'\}\in\mathcal{J}'} \delta_{j_t,j_{t'}},$$

$$Z(I,J)=\sum_{\sigma\in\mathcal{B}(I,J)} \prod_{s\in I} \delta_{i_s,j_{\sigma(s)}} \tag{A79}$$

for subsets $I$ and $J$ of $\{1,\ldots,n\}$. We will use the following obvious relations:

$$X(I)=\sum_{v\in I-\{u\}} \delta_{i_u,i_v}X(I-\{u,v\}) \quad (u\in I \text{ fixed}), \tag{A80}$$

$$Y(J)=\sum_{v\in J-\{u\}} \delta_{j_u,j_v}Y(J-\{u,v\}) \quad (u\in J \text{ fixed}), \tag{A81}$$

$$Z(I,J)=\sum_{v\in J} \delta_{i_u,j_v}Z(I-\{u\},J-\{v\}) \quad (u\in I \text{ fixed}). \tag{A82}$$

With this notation, Eq. (A10) is written

$$\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}=\sum_{0\leqslant k\leqslant n/2} f_{n,k}\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}} \sum_{\substack{I\subset\{1\cdots n\}\\|I|=2k}} \times X(I)Z(\mathbf{C}I,\mathbf{C}J)Y(J). \tag{A83}$$

The sum on the subsets $I\subset\{1,\ldots,n\}$ is decomposed according to the four possible cases $(\emptyset,\{1\},\{2\},\{1,2\})$ of intersection of $I$ with the subset $\{1,2\}$:

$$\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}=\sum_{0\leqslant k\leqslant n/2} f_{n,k}\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\times\Bigg\{\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k}} X(I)Z(\mathbf{C}'I\cup\{1,2\},\mathbf{C}J)+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-1}} X(I\cup\{1\})Z(\mathbf{C}'I\cup\{2\},\mathbf{C}J)+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-1}} X(I\cup\{2\})Z(\mathbf{C}'I\cup\{1\},\mathbf{C}J)+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-2}} X(I\cup\{1,2\})Z(\mathbf{C}'I,\mathbf{C}J)\Bigg\}Y(J). \tag{A84}$$

In this formula, the summation variable $I$ is obtained from the one of (A80) by possibly removing the elements 1 and 2. Furthermore, $\mathbf{C}'I$ is the complementary set in $\{3,...,n\}$ of the subset $I$, while $\mathbf{C}J$ is as before the complementary set of $J$ in $\{1,...,n\}$. Making the indices $i_1$ and $i_2$ explicit is then effected by the following obvious formulas (written for subsets $I\subset\{3,...,n\}$ and $J\subset\{1,...,n\}$), which can also be deduced from Eqs. (A80) and (A82):

$$X(I\cup\{1\})=\sum_{u\in I}\delta_{i_1 i_u}X(I-\{u\}),$$

$$X(I\cup\{2\})=\sum_{u\in I}\delta_{i_2 i_u}X(I-\{u\}),$$

$$X(I\cup\{1,2\})=\delta_{i_1 i_2}X(I)+\sum_{\substack{u,v\in I\\u\neq v}}\delta_{i_1 i_u}\delta_{i_2 i_v}X(I-\{u,v\}),$$

$$Z(I\cup\{1\},J)=\sum_{v\in J}\delta_{i_1 j_v}Z(I,J-\{v\}),$$

$$Z(I\cup\{2\},J)=\sum_{v\in J}\delta_{i_2 j_v}Z(I,J-\{v\}),$$

$$Z(I\cup\{1,2\},J)=\sum_{\substack{u,v\in J\\u\neq v}}\delta_{i_1 j_u}\delta_{i_2 j_v}Z(I,J-\{u,v\}). \tag{A85}$$

One obtains

$$\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}=\sum_{0\leqslant k\leqslant n/2} f_{n,k}\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\Bigg\{\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k}}\sum_{\substack{u,v\notin J\\u\neq v}}\delta_{i_1 j_u}\delta_{i_2 j_v}X(I)Z(\mathbf{C}'I,\mathbf{C}J-\{u,v\})+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-1}}\sum_{u\in I}\sum_{v\notin J}\delta_{i_1 i_u}\delta_{i_2 i_v}\times X(I-\{u\})Z(\mathbf{C}'I,\mathbf{C}J-\{v\})+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-1}}\sum_{u\in I}\sum_{v\notin J}\delta_{i_2 i_u}\delta_{i_1 j_v}X(I-\{u\})Z(\mathbf{C}'I,\mathbf{C}J-\{v\})+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-2}}\delta_{i_1 i_2}X(I)Z(\mathbf{C}'I,\mathbf{C}J)+\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-2}}\sum_{\substack{u,v\in I\\u\neq v}}\delta_{i_1 i_u}\delta_{i_2 i_v}X(I-\{u,v\})Z(\mathbf{C}'I,\mathbf{C}J)\Bigg\}Y(J). \tag{A86}$$

It is now straightforward to contract the indices $i_1$ and $i_2$:

$$\sum_{i_1,i_2=1}^{D}\delta_{i_1 i_2}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}=\sum_{0\leqslant k\leqslant n/2} f_{n,k}\Bigg\{\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k}}\sum_{\substack{u,v\notin J\\u\neq v}}\delta_{j_u j_v}X(I)Z(\mathbf{C}'I,\mathbf{C}J-\{u,v\})Y(J)+2\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-1}}\sum_{u\in I}\sum_{v\notin J}\delta_{i_u j_v}X(I-\{u\})Z(\mathbf{C}'I,\mathbf{C}J-\{v\})Y(J)+D\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-2}}X(I)Z(\mathbf{C}'I,\mathbf{C}J)Y(J)+\sum_{\substack{J\subset\{1\cdots n\}\\|J|=2k}}\sum_{\substack{I\subset\{3\cdots n\}\\|I|=2k-2}}\sum_{\substack{u,v\in I\\u\neq v}}\delta_{i_u i_v}\times X(I-\{u,v\})Z(\mathbf{C}'I,\mathbf{C}J)Y(J)\Bigg\}. \tag{A87}$$

Next we rewrite the first, second, and last terms in the braces in the same form as the third one. Let us transform a partial sum in the *first* term as follows:

$$\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k}}\sum_{\substack{u,v\notin J\\ u\neq v}}\delta_{j_uj_v}Z(\mathbf{C}'I,\mathbf{C}J-\{u,v\})Y(J)$$
$$=\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k+2}}\sum_{\substack{u,v\in J\\ u\neq v}}\delta_{j_uj_v}Z(\mathbf{C}'I,\mathbf{C}J)Y(J-\{u,v\}). \tag{A88}$$

The first sum is, at fixed $u$ and $v$, over the subsets $J$ that contain $u$ and $v$, and the second sum is over the subsets $J$ that do not contain $u$ or $v$. To each subset of the first kind corresponds the subset of the second kind obtained by removing $u$ and $v$, and the original first kind of subset is recovered by including back $u$ and $v$. This proves Eq. (A88). The sum over $u$ and $v$ in the right-hand side of Eq. (A88) is then calculated. By summing Eq. (A81) over $u\in J$, we have

$$\sum_{\substack{u,v\in J\\ u\neq v}}\delta_{j_uj_v}Y(J-\{u,v\})=|J|Y(J). \tag{A89}$$

Combining Eq. (A89) (with $|J|=2k+2$) and Eq. (A88), we have

$$\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k}}\sum_{\substack{u,v\notin J\\ u\neq v}}\delta_{j_uj_v}Z(\mathbf{C}'I,\mathbf{C}J-\{u,v\})Y(J)$$
$$=2(k+1)\sum_{\substack{J'\subset\{1\cdots n\}\\ |J'|=2k+2}}Z(\mathbf{C}'I,\mathbf{C}J')Y(J'). \tag{A90}$$

Let us transform a partial sum in the *second* term as follows:

$$\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-1}}\sum_{u\in I}\sum_{v\notin J}\delta_{i_uj_v}X(I-\{u\})Z(\mathbf{C}'I,\mathbf{C}J-\{v\})$$
$$=\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-2}}\sum_{u\in\mathbf{C}'I}\sum_{v\notin J}\delta_{i_uj_v}X(I)Z(\mathbf{C}'I-\{u\},\mathbf{C}J-\{v\}). \tag{A91}$$

The first sum is, at fixed $u\in\{3,...,n\}$, over the subsets $I$ that contain $u$, and the second sum is over the subsets $I$ that do not contain $u$. The formula (A91) results from the fact that the two kinds of subsets are put in bijective correspondence by removing or including $u$. The sum over $u$ and $v$ in the right-hand side of Eq. (A91) is then calculated. By summing Eq. (A82) over $u\in I$, we have

$$\sum_{u\in I}\sum_{v\in J}\delta_{i_uj_v}Z(I-\{u\},J-\{v\})=|I|Z(I,J). \tag{A92}$$

Combining Eq. (A92) (with $I\to\mathbf{C}'I$, $J\to\mathbf{C}J$, $|I|\to n-2k$) and Eq. (A91), we have

$$\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-1}}\sum_{u\in I}\sum_{v\notin J}\delta_{i_uj_v}X(I-\{u\})Z(\mathbf{C}'I,\mathbf{C}J-\{v\})$$
$$=(n-2k)\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-2}}X(I)Z(\mathbf{C}'I,\mathbf{C}J). \tag{A93}$$

For the *last* term, the sum over $u$ and $v$ is directly calculated by summing Eq. (A80) over $u\in I$; we have

$$\sum_{\substack{u,v\in I\\ u\in v}}\delta_{i_ui_v}X(I-\{u,v\})=|I|X(I). \tag{A94}$$

Using Eqs. (A90), (A93), (A94), the three last terms in Eq. (A87) combine, with a coefficient

$$2(n-2k)+D+2(k-1)=2(n-k+D/2-1), \tag{A95}$$

and formula (A87) becomes

$$\sum_{i_1,i_2=1}^{D}\delta_{i_1i_2}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$$
$$=\sum_{0\leqslant k\leqslant n/2-1}f_{n,k}2(k+1)$$
$$\times\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k}}\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k+2}}X(I)Z(\mathbf{C}'I,\mathbf{C}J)Y(J)$$
$$+\sum_{1\leqslant k\leqslant n/2}f_{n,k}2(n-k+D/2-1)$$
$$\times\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-2}}\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k}}X(I)Z(\mathbf{C}'I,\mathbf{C}J)Y(J). \tag{A96}$$

We have suppressed zero terms in the $k$ sums: in the first sum, the existence of $J\subset\{1,...,n\}$ with $|J|=2k+2$ needs $2k\leqslant n-2$, and in the second sum, the existence of $I$ with $|I|=2k-2$ needs $2k\geqslant 2$. This adjustment of the summation bounds is important because now, after the change of variable $k\to k-1$, the first sum combines exactly with the second one:

$$\sum_{i_1,i_2=1}^{D}\delta_{i_1i_2}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}$$
$$=2\sum_{1\leqslant k\leqslant n/2}[kf_{n,k-1}+(n-k+D/2-1)f_{n,k}]$$
$$\times\sum_{\substack{I\subset\{3\cdots n\}\\ |I|=2k-2}}\sum_{\substack{J\subset\{1\cdots n\}\\ |J|=2k}}X(I)Z(\mathbf{C}'I,\mathbf{C}J)Y(J). \tag{A97}$$

We find thus that, with the trial formula (A10), the tracelessness condition 2 takes the form of the following recurrence relation for the coefficients $f_{n,k}$:

$$kf_{n,k-1}+(n-k+D/2-1)f_{n,k}=0 \quad (1\leqslant k\leqslant n/2), \tag{A98}$$

and this recurrence relation, with the initial condition (A78), uniquely determines the $f_{n,k}$:

$$f_{n,k}=(-1)^k\frac{k!\Gamma(n-k+D/2-1)}{n!\Gamma(n+D/2-1)} =(-1)^k\frac{1}{n!}\binom{n+D/2-1}{k}^{-1} \tag{A99}$$

where the second expression is defined for all required values $D\geqslant 1$, $n\geqslant 0$, $0\leqslant k\leqslant n/2$, while the more explicit first one is ambiguous $(\infty/\infty)$ for $D=2$, $n=0$, $k=0$.

This result can be expressed with factorials, directly in the case of $D$ even and by use of the duplication formula of the $\Gamma$ function in the case of $D$ odd:

$$f_{n,k}=(-1)^k\frac{k!(n-k+D/2-2)!}{n!(n+D/2-2)!} \quad (D \text{ even}) \tag{A100}$$

$$f_{n,k}=(-1)^k 2^{-2k}\frac{k!(n+D/2-3/2)!(2n-2k+D-3)!}{n!(n-k+D/2-3/2)!(2n+D-3)!} \times(D \text{ odd}). \tag{A101}$$

For the spatial dimension $D=3$, the expression (A11) of $f_{n,k}$ is recovered.

### Calculation of the particular matrix elements from the expression of the projector

By using the explicit expression (A10) of the projector, it is of course possible to compute any matrix element needed. As a simple example, we present here the calculation of the particular matrix elements (A3).

We may do all the calculations for a particle at rest $[v=(1,\vec{0})]$, and write the final result covariantly in term of the four-vectors $v,v_i,v_f$. Then we have

$$v_f^{\mu_1}\cdots v_f^{\mu_n}\Pi_{\mu_1\cdots\mu_n;\nu_1\cdots\nu_n}v_i^{\nu_1}\cdots v_i^{\nu_n} =\sum_{i_1,\cdots i_n,j_1,\cdots j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n}. \tag{A102}$$

The calculation of this from Eq. (A10) simply amounts to the substitutions

$$\delta_{i_r i_{r'}}\to(\vec{v}_f)^2, \quad \delta_{i_s j_{\sigma(s)}}\to(\vec{v}_f\cdot\vec{v}_i), \quad \delta_{j_t j_{t'}}\to(\vec{v}_i)^2, \tag{A103}$$

and we have

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} =\sum_{0\leqslant k\leqslant n/2} f_{n,k}\sum_{\substack{I,J\subset\{1\cdots n\}\\ |I|=|J|=2k}}\ \sum_{\mathcal{J}\in\mathcal{P}_2(I)}\ \sum_{\sigma\in\mathcal{B}(\mathbf{C}I,\mathbf{C}J)}\ \sum_{\mathcal{J}'\in\mathcal{P}_2(J)} \times(\vec{v}_f)^{2k}(\vec{v}_f\cdot\vec{v}_i)^{n-2k}(\vec{v}_i)^{2k}. \tag{A104}$$

The summand is independent of the summation variables $I,J,\mathcal{J},\sigma,\mathcal{J}'$, so that we just have to count the number of values they take. As is well known, the number of subsets of cardinality $q$ in a set of cardinality $n$ is given by the binomial coefficient $\binom{n}{q}=n!/q!(n-q)!$, and the number $|\mathcal{B}(X,Y)|$ of bijections of a set $X$ on a set $Y$ is $|X|!$ if $|X|=|Y|$ (and 0 else). The number $|\mathcal{P}_2(X)|$ of partitions of a set $X$ by two-element subsets is $(2k)!/2^k k!$ when $|X|=2k$. This last number is easily obtained by first considering the set of decompositions of $X$ into ordered $k$-uples of ordered pairs. The number of such $k$-uples is the same $(2k)!$ as the number of permutations of $X$, and when the orderings (of the $k$-uples and of the pairs) are disregarded, there are $2^k k!$ $k$-uples giving each partition by two-element subsets. Therefore the numbers of values taken by the summation variables in Eq. (A104) are

$$\frac{n!}{(2k)!(n-2k)!} \quad \text{for } I \text{ and for } J,$$

$$\frac{(2k)!}{2^k k!} \quad \text{for } \mathcal{J} \text{ and for } \mathcal{J}',$$

$$(n-2k)! \quad \text{for } \sigma;$$

and from Eq. (A104) we have

$$\begin{aligned}\sum_{i_1,\cdots i_n,j_1,\cdots j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} &=\sum_{0\leqslant k\leqslant n/2} f_{n,k}\left(\frac{n!}{(2k)!(n-2k)!}\right)^2\left(\frac{(2k)!}{2^k k!}\right)^2 \times(n-2k)!(\vec{v}_f)^{2k}(\vec{v}_f\cdot\vec{v}_i)^{n-2k}(\vec{v}_i)^{2k}\\ &=\sum_{0\leqslant k\leqslant n/2} 2^{-2k}\frac{(n!)^2}{(k!)^2(n-2k)!}f_{n,k}(\vec{v}_f)^{2k} \times(\vec{v}_f\cdot\vec{v}_i)^{n-2k}(\vec{v}_i)^{2k}.\end{aligned} \tag{A105}$$

With $(\vec{v}_f)^2,(\vec{v}_f\cdot\vec{v}_i),(\vec{v}_i)^2$ written in covariant form using Eq. (A41), we recover Eq. (A5) with $C_{n,k}$ given by

$$C_{n,k}=(-1)^k 2^{-2k}\frac{n!}{(k!)^2(n-2k)!}\binom{n+D/2-2}{k}^{-1} \tag{A106}$$

for an arbitrary spatial dimension $D$. A compact expression like Eq. (A3) is obtained from Eqs. (A5) and (A106) by introducing the Gegenbauer polynomials $C_n^\lambda$, which can be defined by the generating functions

$$\sum_{n\geqslant 0} C_n^{\lambda}(x)t^n=\frac{1}{(1-2xt+t^2)^{\lambda}} \tag{A107}$$

and have the following expressions:

$$C_n^{\lambda}(x)=\sum_k (-1)^k\binom{n-k}{k}\binom{n-k+\lambda-1}{n-k}(2x)^{n-2k}, \tag{A108}$$

$$C_n^{\lambda}(x)=\sum_k (-1)^k\binom{k+\lambda-1}{k}\binom{n+2\lambda-1}{n-2k}x^{n-2k}\times(1-x^2)^k. \tag{A109}$$

Indeed, Eq. (A106) becomes

$$C_{n,k}=(-1)^k 2^{-2k}\binom{n+D/2-2}{n}^{-1}\binom{n-k}{k}\binom{n-k+D/2-2}{n-k}, \tag{A106$'$}$$

and using Eq. (A108) we have

$$\sum_{0\leqslant k\leqslant n/2} C_{n,k}x^{n-2k}=\frac{1}{2^n}\binom{n+D/2-2}{n}^{-1}C_n^{D/2-1}(x), \tag{A110}$$

obtaining

$$\sum_{i_1,\cdots i_n,j_1,\cdots j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} =\frac{1}{2^n}\binom{n+D/2-2}{n}^{-1}|\vec{v}_f|^n|\vec{v}_i|^n C_n^{D/2-1}\left(\frac{\vec{v}_f\cdot\vec{v}_i}{|\vec{v}_f||\vec{v}_i|}\right). \tag{A111}$$

When $D=3$, expression (A3) is recovered from Eq. (A111) using $C_n^{1/2}(x)=P_n(x)$ and $\binom{n-1/2}{n}=2^{-2n}(2n)!/(n!)^2$. Using in Eq. (A111) the expression (A109) of $C_n^{\lambda}(x)$, one obtains the expression (A6) with $C'_{n,k}$ given by

$$C'_{n,k}=(-1)^k\frac{1}{2^n}\binom{n+D/2-2}{n}^{-1}\binom{k+D/2-2}{k}\binom{n+D-3}{n-2k}. \tag{A112}$$

Notice that the expression (A106$'$) is ambiguous for $D=2$, $n\geqslant 1$ [while Eq. (A106) is defined for all required values $D\geqslant 1$, $n\geqslant 0$, $0\leqslant k\leqslant n/2$], and the ambiguity propagates to Eqs. (A111) and (A112). If we want the case $D=2$ to be included in our formulas, we may use, instead of the Gegenbauer polynomials, the Jacobi polynomials simply related to them by

$$C_n^{\lambda}(x)=\frac{(2\lambda)_n}{(\lambda+1/2)_n}P_n^{(\lambda-1/2,\lambda-1/2)}(x), \tag{A113}$$

which have the following expressions:

$$P_n^{(\alpha,\alpha)}(x)=2^{-n}\sum_k (-1)^k\binom{n+\alpha}{k}\binom{2n-2k+2\alpha}{n-2k}x^{n-2k}, \tag{A114}$$

$$P_n^{(\alpha,\alpha)}(x)=\sum_k (-1)^k 2^{-2k}\binom{n-k}{k}\binom{n+\alpha}{n-k}x^{n-2k}(1-x^2)^k. \tag{A115}$$

Rewriting Eq. (A106) as

$$C_{n,k}=(-1)^k\binom{2n+D-3}{n}^{-1}\binom{n+D/2-3/2}{k}\times\binom{2n-2k+D-3}{n-2k} \tag{A106$''$}$$

from Eq. (A114) we see that

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} =2^n\binom{2n+D-3}{n}^{-1}|\vec{v}_f|^n|\vec{v}_i|^n P_n^{(D-3)/2,(D-3)/2)}\left(\frac{\vec{v}_f\cdot\vec{v}_i}{|\vec{v}_f||\vec{v}_i|}\right) \tag{A111$'$}$$

and using Eq. (A115) we obtain

$$C'_{n,k}=(-1)^k 2^{n-2k}\binom{2n+D-3}{n}^{-1}\binom{n-k}{k}\binom{n+D/2-3/2}{n-k}. \tag{A112$'$}$$

The expression (A106$''$) is ambiguous only for $D=1$, $n=1$, and so are Eqs. (A111$'$) and (A112$'$).

Finally, let us consider the case $D=1$. It is in fact trivial. All $n$-rank tensor spaces ($n\geqslant 0$) are of dimension 1. All $n$-rank tensors are symmetric. The subspace of $n$-rank symmetric traceless tensors is the whole space when $n=0$ or 1, and is the *zero subspace* when $n\geqslant 2$. Therefore, the projector on this subspace is the identity operator when $n=0$ or 1 (as for any dimension $D$), and is, when $n\geqslant 2$, the *zero operator*.

To see how this particular case is obtained with our results, we may apply Eq. (A111), which is well defined for $D=1$. First, due to $\vec{v}_f\cdot\vec{v}_i=|\vec{v}_f||\vec{v}_i|\mathrm{sgn}(\vec{v}_f\cdot\vec{v}_i)$, Eq. (A111) becomes

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} =\frac{1}{2^n}\binom{n-3/2}{n}^{-1}C_n^{-1/2}(1)(\vec{v}_f\cdot\vec{v}_i)^n \tag{A116}$$

(where the sum has only one term). Next, $C_n^{-1/2}(1)$ is given by Eq. (A109):

$$C_n^{-1/2}(1)=\binom{n-2}{n}=(-1)^n\binom{1}{n}=\delta_{n,0}-\delta_{n,1}. \tag{A117}$$

Then we have

$$\frac{1}{2^n}\binom{n-3/2}{n}^{-1}\bigg|_{n=0}=1,\quad \frac{1}{2^n}\binom{n-3/2}{n}^{-1}\bigg|_{n=1}=-1, \tag{A118}$$

so that Eq. (A116) becomes

$$\sum_{i_1,\ldots,i_n,j_1,\ldots,j_n} v_f^{i_1}\cdots v_f^{i_n}\Pi_{i_1,\ldots,i_n;j_1,\ldots,j_n}v_i^{j_1}\cdots v_i^{j_n} = \delta_{n,0}+(\vec{v}_f\cdot\vec{v}_i)\,\delta_{n,1}, \tag{A119}$$

and, as expected, this vanishes for $n\geqslant 2$.

## APPENDIX B: MANIFESTLY COVARIANT DERIVATION OF BJORKEN AND URALTSEV SUM RULES

In this appendix we give a manifestly covariant derivation of the Bjorken and Uraltsev SRs using the states and currents considered by Uraltsev [6]. He considers the fourth component of the vector current, and initial and final $B^*$ states, allowing for spin flip transitions, i.e., with our notation, he takes $\Gamma_1=\Gamma_2=\gamma^0$, the initial and final states $\mathcal{B}_i=B^{*(\lambda_i)}(1,\mathbf{0})$, $\mathcal{B}_f=B^{*(\lambda_f)}(v_f^0,\mathbf{v}_f)$, and performs an expansion for small velocities. In the covariant language adopted here, the case he considers is

$$\Gamma_1=\Gamma_2=\not{v}_i,$$

$$\mathcal{B}_i=P_{i+}\not{\varepsilon}_i,\quad \mathcal{B}_f=P_{f+}\not{\varepsilon}_f. \tag{B1}$$

We realize that this case does not present the symmetry of the simple choice of Secs. III and IV, since both currents, projected in the $v_i$ direction, appear in a nonsymmetric way relative to the initial and final states, which have four-velocities $v_i$ and $v_f$. This aspect, plus the $B^*$ polarization, complicates the calculation in a considerable way, since then all states $2^+_{3/2}$, $1^+_{3/2}$, $0^+_{1/2}$, and $1^+_{1/2}$ contribute. We give now the covariant version of the Uraltsev calculation.

After a good deal of algebra, the RHS of the general SR (38) becomes for the choice (B1),

$$\begin{aligned}R(w_i,w_f,w_{if})=\xi(w_{if})\{&(\varepsilon_i\cdot\varepsilon_f)(w_i+w_f)-(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')\\&-(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v')-(2w_i+1)[(\varepsilon_i\cdot\varepsilon_f)\\&\times(w_{if}+1)-(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)]\}\end{aligned} \tag{B2}$$

while the contribution of the different intermediate states is given by

$$L(0^-_{1/2})=0, \tag{B3}$$

$$\begin{aligned}L(1^-_{1/2})=\{&-(w_i+1)[(\varepsilon_i\cdot\varepsilon_f)(w_{if}+w_i)-(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v_f)+(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v')]+(\varepsilon_i\cdot v')[(\varepsilon_f\cdot v_i)(2w_i-w_f+1)+(\varepsilon_f\cdot v')\\&\times(w_{if}-1)]\}\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f),\end{aligned} \tag{B4}$$

$$\begin{aligned}L(2^+_{3/2})=\Big\{&\frac{1}{2}(w_{if}-w_fw_i)(w_i+1)[(\varepsilon_i\cdot\varepsilon_f)w_{if}-(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v_f)+(\varepsilon_i\cdot\varepsilon_f)w_i+(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v')]-\frac{1}{2}(w_{if}-w_fw_i)(\varepsilon_i\cdot v')\\&\times[(\varepsilon_f\cdot v_i)w_i+(\varepsilon_f\cdot v_i)w_i-(\varepsilon_f\cdot v')]+\frac{1}{6}(-2-2w_i-2w_f-3w_{if}+4w_iw_f)(\varepsilon_i\cdot v')[(\varepsilon_f\cdot v_i)(1-w_f)\\&+(\varepsilon_f\cdot v')w_{if}]-\frac{1}{2}[(\varepsilon_i\cdot v_f)(w_i+1)-(\varepsilon_i\cdot v')w_{if}](\varepsilon_f\cdot v')-\frac{1}{2}w_i[(\varepsilon_i\cdot v_f)(w_i+1)-(\varepsilon_i\cdot v')w_{if}][(\varepsilon_f\cdot v_i)(1-w_f)\\&+(\varepsilon_f\cdot v')w_{if}]+\frac{1}{2}w_f(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')+w_i[(\varepsilon_i\cdot v_f)(w_i+1)-(\varepsilon_i\cdot v')(v_i\cdot v_f)](\varepsilon_f\cdot v_i)-w_iw_f(\varepsilon_i\cdot v')\\&\times(\varepsilon_f\cdot v_i)\Big\}3\sum_n \tau^{(n)}_{3/2}(w_i)\tau^{(n)}_{3/2}(w_f),\end{aligned} \tag{B5}$$

$$\begin{aligned}L(1^+_{3/2})=\Big\{&-\frac{1}{6}(1+w_i)(1+w_f)\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\gamma^\sigma\not{v}'\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\gamma_\sigma\not{v}_f\gamma_5]+\frac{1}{6}(1+w_i)\\&\times(1+w_f)\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\gamma_\sigma\gamma_5]-\frac{1}{2}(1+w_i)\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\not{v}_f\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\not{v}_f\gamma_5]\Big\}\\&\times 3\sum_n \tau^{(n)}_{3/2}(w_i)\tau^{(n)}_{3/2}(w_f),\end{aligned} \tag{B6}$$

$$L(0^+_{1/2})=(\varepsilon_i\cdot v')[(\varepsilon_f\cdot v_i)(1-w_f)+(\varepsilon_f\cdot v')(v_i\cdot v_f)]4\sum_n \tau^{(n)}_{1/2}(w_i)\tau^{(n)}_{1/2}(w_f), \tag{B7}$$

$$L(1^+_{1/2})=\left\{-\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}_f\gamma_\sigma\gamma_5]+\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\gamma_\sigma\gamma_5]\right\}4\sum_n \tau^{(n)}_{1/2}(w_i)\tau^{(n)}_{1/2}(w_f), \tag{B8}$$

$$L(2^-_{3/2})=\frac{1}{2}\left\{(w_{if}-w_iw_f)(w_i+1)\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}_f\gamma_\sigma\gamma_5]-(w_{if}-w_iw_f)\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\gamma_\sigma\gamma_5]\right.$$
$$\left.+w_i\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\not{v}_f\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\not{v}_f\gamma_5]\right\}3\sum_n \sigma^{(n)}_{3/2}(w_i)\sigma^{(n)}_{3/2}(w_f),$$

$$L(1^-_{3/2})=\left\{-\frac{1}{6}(w_i-1)(w_f-1)(w_i+1)[(\varepsilon_i\cdot\varepsilon_f)(w_{if}+w_i)+(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v')-(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v_f)]+\frac{1}{6}(w_i-1)(w_f-1)\right.$$
$$\times(\varepsilon_i\cdot v')[2(\varepsilon_f\cdot v_i)w_i-(\varepsilon_f\cdot v')]+\frac{1}{6}(1-9w_{if}+4w_iw_f+2w_i+2w_f)(\varepsilon_i\cdot v')[(\varepsilon_f\cdot v_i)(1-w_f)+(\varepsilon_f\cdot v')w_{if}]$$
$$+\frac{1}{2}(w_f-1)(\varepsilon_i\cdot v')[2w_i(\varepsilon_f\cdot v_i)-(\varepsilon_f\cdot v')]+\frac{1}{2}(w_i-1)[(\varepsilon_i\cdot v_f)(w_i+1)-(\varepsilon_i\cdot v')w_{if}]$$
$$\left.\times[(\varepsilon_f\cdot v')w_{if}+(\varepsilon_f\cdot v_i)(1-w_f)]\right\}3\sum_n \sigma^{(n)}_{3/2}(w_i)\sigma^{(n)}_{3/2}(w_f).$$

From

$$\frac{1}{4}\mathrm{Tr}[\not{a}\not{b}\not{c}\not{d}\gamma_5]=-i\varepsilon_{\mu\nu\rho\sigma}a^\mu b^\nu c^\rho d^\sigma, \tag{B9}$$

one can express the contributions $L(1^+_{3/2})$, $L(1^+_{1/2})$, and $L(2^-_{3/2})$ in terms of scalar products. Indeed, the product of two tensors $\varepsilon_{\mu\nu\rho\sigma}$ even *noncontracted* can be expressed in terms of the tensor $g_{\mu\nu}$:

$$\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{\mu'\nu'\rho'\sigma'}=-\det(g_{\alpha\alpha'})\quad(\alpha=\mu,\nu,\rho,\sigma,\ \ \alpha'=\mu',\nu',\rho',\sigma'), \tag{B10}$$

$$g^{\mu\mu'}\varepsilon_{\mu\nu\rho\sigma}\varepsilon_{\mu'\nu'\rho'\sigma'}=-\det(g_{\alpha\alpha'})\quad(\alpha=\nu,\rho,\sigma,\ \ \alpha'=\nu',\rho',\sigma'). \tag{B11}$$

From these relations one obtains, for the traces involved in $L(1^+_{3/2})$, $L(1^+_{1/2})$, and $L(2^-_{3/2})$,

$$\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}_f\gamma_\sigma\gamma_5]=-(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')+w_i(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)-(w_{if}w_i-w_f)(\varepsilon_i\cdot\varepsilon_f), \tag{B12}$$

$$\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\gamma^\sigma\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\gamma_\sigma\gamma_5]=-(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')+w_i(\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)-(w_i^2-1)(\varepsilon_i\cdot\varepsilon_f), \tag{B13}$$

$$\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_i\not{v}'\not{v}_f\gamma_5]\frac{1}{4}\mathrm{Tr}[\not{v}_i\not{\varepsilon}_f\not{v}'\not{v}_f\gamma_5]=(w_{if}^2-1)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')-(w_{if}w_f-w_i)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)$$
$$-(w_{if}w_i-w_f)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')+(w_{if}-w_iw_f)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)$$
$$+(2w_{if}w_iw_f-w_{if}^2-w_i^2-w_f^2+1)(\varepsilon_i\cdot\varepsilon_f). \tag{B14}$$

From the latter expressions (B12)–(B14) and from (B2)–(B8) one gets finally for Eq. (38)

$$\{-(w_i+1)(w_{if}+w_i)(\varepsilon_i\cdot\varepsilon_f)+(w_i+1)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)+(w_i-w_f)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)+(w_{if}-1)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')\}$$
$$\times\sum_n \xi^{(n)}(w_i)\xi^{(n)}(w_f)+\{-(w_i+1)(4w_fw_iw_{if}+2w_fw_i^2-w_i^2-w_f^2-2w_{if}w_i-3w_{if}^2+1)(\varepsilon_i\cdot\varepsilon_f)$$
$$+(w_i+1)(4w_iw_f-3w_{if}+w_i)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)-(w_i+1)(w_f+1)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')+[(3w_{if}-2w_iw_f+w_f)(w_f-w_i)$$
$$-(w_i+1)^2](\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)+(2w_iw_f-3w_{if}-w_i-w_f-1)(w_{if}-1)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')\}\sum_n \tau^{(n)}_{3/2}(w_i)\tau^{(n)}_{3/2}(w_f)$$

$$+4\{(w_{if}w_i-w_i^2-w_f+1)(\varepsilon_i\cdot\varepsilon_f)-w_i(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)+(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')+(1+w_i-w_f)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)$$
$$+(w_{if}-1)(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')\}\sum_n \tau_{1/2}^{(n)}(w_i)\tau_{1/2}^{(n)}(w_f)+\{(w_i+2w_iw_f-3w_f^2w_i-w_i^3-2w_fw_i^3-2w_{if}+2w_fw_{if}+2w_i^2w_{if}$$
$$+4w_fw_i^2w_{if}-3w_iw_{if}^2)(\varepsilon_i\cdot\varepsilon_f)+(-1+w_f+w_i^2-4w_fw_i^2+3w_iw_{if})(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v_f)+3(w_iw_f-w_{if})(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')$$
$$+(w_f-w_f^2-w_i+3w_iw_f-2w_f^2w_i+w_i^2+2w_fw_i^2-3w_{if}+3w_fw_{if}-3w_iw_{if})(\varepsilon_f\cdot v_i)(\varepsilon_i\cdot v')$$
$$+(w_{if}-1)(-1+w_i+w_f+2w_iw_f-3w_{if})(\varepsilon_i\cdot v')(\varepsilon_f\cdot v')\}\sum_n \sigma_{3/2}^{(n)}(w_i)\sigma_{3/2}^{(n)}(w_f)+\cdots$$
$$=\xi(w_{if})\{[(w_i+w_f)-(2w_i+1)(w_{if}+1)](\varepsilon_i\cdot\varepsilon_f)+(2w_i+1)(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v_i)-(\varepsilon_i\cdot v_f)(\varepsilon_f\cdot v')-(\varepsilon_i\cdot v')(\varepsilon_f\cdot v_i)\}. \tag{B15}$$

This expression is considerably more complicated than Eq. (48), which readily gives the Uraltsev SR. We can choose the particular polarizations

$$\varepsilon_i^{(1)}=\frac{v_f-w_{if}v_i}{\sqrt{w_{if}^2-1}},\quad \varepsilon_f^{(1)}=\frac{v_i-w_{if}v_f}{\sqrt{w_{if}^2-1}},$$
$$\varepsilon_i^{(2)}=\frac{v'-w_iv_i}{\sqrt{w_i^2-1}},\quad \varepsilon_f^{(2)}=\frac{v'-w_fv_f}{\sqrt{w_f^2-1}} \tag{B16}$$

that satisfy $\varepsilon_i^2=-1$, $\varepsilon_i\cdot v_i=0$, $\varepsilon_f^2=-1$, $\varepsilon_f\cdot v_f=0$.

We can consider the following different cases:

$$(1)\quad \varepsilon_i=\varepsilon_i^{(1)},\quad \varepsilon_f=\varepsilon_f^{(1)},$$
$$(2)\quad \varepsilon_i=\varepsilon_i^{(2)},\quad \varepsilon_f=\varepsilon_f^{(1)},$$
$$(3)\quad \varepsilon_i=\varepsilon_i^{(1)},\quad \varepsilon_f=\varepsilon_f^{(2)},$$
$$(4)\quad \varepsilon_i=\varepsilon_i^{(2)},\quad \varepsilon_f=\varepsilon_f^{(2)}. \tag{B17}$$

These four different cases exhaust the number of independent SRs in the case under consideration, characterized by Eq. (B1).

That there are only four independent SRs can be seen by the following argument. If, in the general SR (B15), we make the replacements (the sum over $\alpha$ denotes the sum over the different polarizations)

$$\varepsilon_i^\mu\to\sum_\alpha \varepsilon_i^{(\alpha)\mu}\varepsilon_i^{(\alpha)*\rho}=g^{\rho\mu}-v_i^\rho v_i^\mu,$$
$$\varepsilon_f^\nu\to\sum_\alpha \varepsilon_f^{(\alpha)\nu}\varepsilon_f^{(\alpha)*\sigma}=g^{\sigma\nu}-v_f^\sigma v_i^\nu, \tag{B18}$$

we obtain a set of tensorial identities, which depend only on $v_i$, $v_f$, and $v'$:

$$X^{\rho\sigma}(v_i,v_f,v')=0. \tag{B19}$$

From these 16 identities one obtains nine scalar identities saturating all the pairs $v_{i\rho}v_{i\sigma}$, $v_{f\rho}v_{f\sigma}$, $v'_\rho v'_\sigma$, $v_{i\rho}v_{f\sigma}$, etc.,

$$v_{i\rho}v_{i\sigma}X^{\rho\sigma}(v_i,v_f,v')=0,$$
$$\vdots \tag{B20}$$

plus three other scalar identities, identically vanishing,

$$\varepsilon_{\mu\nu\rho\sigma}v_i^\mu v_f^\nu X^{\rho\sigma}(v_i,v_f,v')\equiv 0,$$
$$\vdots \tag{B21}$$

However, among these equations, only four are independent, corresponding to the two nonvanishing products

$$v_{f\rho}(g^{\rho\mu}-v_i^\rho v_i^\mu)=v_f^\mu-w_{if}v_i^\mu,$$
$$v'_\rho(g^{\rho\mu}-v_i^\rho v_i^\mu)=v'^\mu-w_iv_i^\mu, \tag{B22}$$

which must be combined with the other two four-vectors:

$$v_{i\rho}(g^{\rho\mu}-v_f^\rho v_f^\mu)=v_i^\mu-w_{if}v_f^\mu,$$
$$v'_\rho(g^{\rho\mu}-v_f^\rho v_f^\mu)=v'^\mu-w_fv_f^\mu. \tag{B23}$$

Let us now write this SR (B15) for the different cases. We need the following scalar products:

$$\varepsilon_i^{(1)}\cdot v_f=-\frac{w_{if}^2-1}{\sqrt{w_{if}^2-1}},\quad \varepsilon_i^{(1)}\cdot v'=-\frac{w_{if}w_i-w_f}{\sqrt{w_{if}^2-1}},$$
$$\varepsilon_i^{(2)}\cdot v_f=-\frac{w_{if}w_i-w_f}{\sqrt{w_i^2-1}},\quad \varepsilon_i^{(2)}\cdot v'=-\frac{w_i^2-1}{\sqrt{w_i^2-1}},$$
$$\varepsilon_f^{(1)}\cdot v_i=-\frac{w_{if}^2-1}{\sqrt{w_{if}^2-1}},\quad \varepsilon_f^{(1)}\cdot v'=-\frac{w_{if}w_f-w_i}{\sqrt{w_{if}^2-1}},$$

$$\varepsilon_f^{(2)}\cdot v_i = -\frac{w_{if}w_f - w_i}{\sqrt{w_f^2-1}}, \quad \varepsilon_f^{(2)}\cdot v' = -\frac{w_f^2-1}{\sqrt{w_f^2-1}},$$

$$\varepsilon_i^{(1)}\cdot\varepsilon_f^{(1)} = \frac{w_{if}(w_{if}^2-1)}{\sqrt{w_{if}^2-1}\sqrt{w_{if}^2-1}},$$

$$\varepsilon_i^{(2)}\cdot\varepsilon_f^{(1)} = \frac{w_{if}(w_{if}w_i - w_f)}{\sqrt{w_i^2-1}\sqrt{w_{if}^2-1}},$$

$$\varepsilon_i^{(1)}\cdot\varepsilon_f^{(2)} = \frac{w_{if}(w_{if}w_f - w_i)}{\sqrt{w_f^2-1}\sqrt{w_{if}^2-1}},$$

$$\varepsilon_i^{(2)}\cdot\varepsilon_f^{(2)} = \frac{w_{if}w_iw_f - w_i^2 - w_f^2 + 1}{\sqrt{w_{if}^2-1}\sqrt{w_{if}^2-1}}. \tag{B24}$$

Since Eq. (B15) is linear in $\varepsilon_i$ and in $\varepsilon_f$, in deducing the equation for the different cases we can multiply Eq. (B15) by the denominators defining the polarizations in Eq. (B16). We thus obtain from Eq. (B15) four different equations for the different cases (B17).

If, in particular, we make $w_i = w_f = w$, we obtain the following equations, for the different cases considered:

(1) $\varepsilon_i = \varepsilon_i^{(1)},\ \varepsilon_f = \varepsilon_f^{(1)}$:

$$\begin{aligned}&-(w_{if}-1)(1+w-w^2-w_{if}+2ww_{if}+3w^2w_{if}+ww_{if}^2)\sum_n [\xi^{(n)}(w)]^2+(w_{if}-1)(w-2w^2-4w^3+2w^4+2w_{if}+ww_{if}\\&-4w^2w_{if}-6w^4w_{if}+2w_{if}^2+3ww_{if}^2+6w^2w_{if}^2-4w^3w_{if}^2+3ww_{if}^3)\sum_n [\tau_{3/2}^{(n)}(w)]^2-4(w_{if}-1)(w-w^2-w_{if}+3w^2w_{if}-w_{if}^2\\&-ww_{ij}^2)\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w_{if}-1)(1-w-2w^2+2w^4+w_{if}+3ww_{if}-4w^3w_{if}-6w^4w_{if}-3w_{if}^2+ww_{if}^2\\&+10w^2w_{if}^2+4w^3w_{if}^2-3w_{if}^3-3ww_{if}^3)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=-(w_{if}-1)(w_{if}+1)(1+w_{if}+2ww_{if})\xi(w_{if});\end{aligned} \tag{B25}$$

(2) $\varepsilon_i = \varepsilon_i^{(2)},\ \varepsilon_f = \varepsilon_f^{(1)}$:

$$\begin{aligned}&-w(w+1)(w_{if}-1)(w+w_{if})\sum_n [\xi^{(n)}(w)]^2+(w+1)(w_{if}-1)(1-2w^2-2w^4+w_{if}+ww_{if}+2w^2w_{if}-4w^3w_{if}+3ww_{if}^2)\\&\times\sum_n [\tau_{3/2}^{(n)}(w)]^2+4(w-1)(w+1)(w_{if}-1)(1-w+w_{if})\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w-1)(w+1)(w_{if}-1)(2w^2-2w^3\\&-3w_{if}+2ww_{if}+4w^2w_{if}-3w_{if}^2)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=-(w+1)(w_{if}+1)(w_{if}-1)(2w-1)\xi(w_{if});\end{aligned} \tag{B26}$$

(3) $\varepsilon_i = \varepsilon_i^{(1)},\ \varepsilon_f = \varepsilon_f^{(2)}$:

$$\begin{aligned}&-w(w+1)(w_{if}-1)(w+w_{if})\sum_n [\xi^{(n)}(w)]^2+(w+1)(w_{if}-1)(1-2w^2-2w^4+w_{if}+ww_{if}+2w^2w_{if}-4w^3w_{if}+3ww_{if}^2)\\&\times\sum_n [\tau_{3/2}^{(n)}(w)]^2+4(w-1)(w+1)(w_{if}-1)(1-w+w_{if})\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w-1)(w_{if}-1)(-w^2+3w^3-2w^4\\&-3w_{if}-ww_{if}+9w^2w_{if}+w^3w_{if}-3w_{if}^2-3ww_{if}^2)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=-(w+1)(w_{if}+1)(w_{if}-1)(2w-1)\xi(w_{if});\end{aligned} \tag{B27}$$

(4) $\varepsilon_i = \varepsilon_i^{(2)},\ \varepsilon_f = \varepsilon_f^{(2)}$:

$$(w+1)^2(-1+w+w^2-ww_{if})\sum_n [\xi^{(n)}(w)]^2+(w+1)^2(-w-2w^2+4w^3+2w^4+2w_{if}-2ww_{if}-2w^2w_{if}+4w^3w_{if}+3ww_{if}^2)\sum_n [\tau_{3/2}^{(n)}(w)]^2+4(w-1)^2(w+1)(w+w_{if})\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w-1)^2(w+1)(1-2w^2+2w^3-2ww_{if}+4w^2w_{if}-3w_{if}^2)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=(w+1)(-1+3w-w_{if}-3ww_{if}+2w^2w_{if})\xi(w_{if}). \quad \text{(B28)}$$

Let us now consider the SRs that can be obtained without deriving the function $\xi(w_{if})$.

(1) $\varepsilon_i=\varepsilon_i^{(1)},\ \varepsilon_f=\varepsilon_f^{(1)}$. Dividing by $(w_{if}-1)$ and taking the limit $w_{if}\to 1$, one gets

$$-2(w+1)^2\sum_n [\xi^{(n)}(w)]^2-4(w-1)(w+1)^3\times\sum_n [\tau_{3/2}^{(n)}(w)]^2-8(w-1)(w+1)\sum_n [\tau_{1/2}^{(n)}(w)]^2-4(w^2-1)^2\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=-4(w+1). \quad \text{(B29)}$$

(2) $\varepsilon_i=\varepsilon_i^{(2)},\ \varepsilon_f=\varepsilon_f^{(1)}$ and (3) $\varepsilon_i=\varepsilon_i^{(1)},\ \varepsilon_f=\varepsilon_f^{(2)}$. Dividing by $(w_{if}-1)$ and taking the limit $w_{if}\to 1$, one obtains

$$-w(w+1)^2\sum_n [\xi^{(n)}(w)]^2-2(w-1)(w+1)^4\times\sum_n [\tau_{3/2}^{(n)}(w)]^2-4(w-1)(w+1)(w-2)\times\sum_n [\tau_{1/2}^{(n)}(w)]^2+2(w^2-1)^2(3-w)\times\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=-2(w+1)(2w-1). \quad \text{(B30)}$$

(4) $\varepsilon_i=\varepsilon_i^{(2)},\ \varepsilon_f=\varepsilon_f^{(2)}$. Taking the limit $w_{if}\to 1$, one gets

$$(w-1)(w+1)^3\sum_n [\xi^{(n)}(w)]^2+2(w+1)^4(w-1)^2\times\sum_n [\tau_{3/2}^{(n)}(w)]^2+4(w-1)^2(w+1)^2\sum_n [\tau_{1/2}^{(n)}(w)]^2+2(w^2-1)^3\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=2(w+1)^2(w-1). \quad \text{(B31)}$$

From Eqs. (B26), (B27) one gets the two sum rules

$$\frac{w+1}{2}\sum_n [\xi^{(n)}(w)]^2+(w-1)\left\{2\sum_n [\tau_{1/2}^{(n)}(w)]^2+(w+1)^2\sum_n [\tau_{3/2}^{(n)}(w)]^2\right\}+(w+1)(w-1)^2\times\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=1, \quad \text{(B32)}$$

$$w\frac{w+1}{2}\sum_n [\xi^{(n)}(w)]^2+(w-1)\left\{(w+1)^3\sum_n [\tau_{3/2}^{(n)}(w)]^2+2(w-2)\sum_n [\tau_{1/2}^{(n)}(w)]^2\right\}-(w+1)(w-1)^2\times(3-w)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=2w-1. \quad \text{(B33)}$$

The first SR is the Bjorken SR (53).

Eliminating $[(w+1)/2]\Sigma_n|\xi^{(n)}(w)|^2$ between these equations, one obtains

$$(w+1)^2\sum_n |\tau_{3/2}^{(n)}(w)|^2-4\sum_n |\tau_{1/2}^{(n)}(w)|^2-3(w^2-1)\sum_n [\sigma_{3/2}^{(n)}(w)]^2+\cdots=1. \quad \text{(B34)}$$

Equation (B34) is another generalization of the Uraltsev SR for $w\neq 1$; indeed, it reduces to Eq. (57) for $w=1$. Notice that the states $\frac{3}{2}^-$ contribute at order $(w-1)$ to Eq. (B34), while they do not contribute at all to the generalization of Sec. V of the Uraltsev SR for $w\neq 1$ [Eq. (67)]. There is no contradiction: these are two different generalizations, and the difference can be traced back to the fact that the former is obtained from the currents $\{\not{v}_i,\not{v}_i\}$ while the latter is obtained from symmetric currents $\{\not{v}_i\gamma_5,\not{v}_f\gamma_5\}$ relative to the initial and final four-velocities.

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